

# DEFORMATION THEORY OF RIGID-ANALYTIC SPACES

ISAMU IWANARI

**ABSTRACT.** In this paper, we study a deformation theory of rigid analytic spaces. We develop a theory of cotangent complexes for rigid geometry which fits in with our deformations. We then use the complexes to give a cohomological description of infinitesimal deformation of rigid analytic spaces. Moreover, we will prove an existence of a formal versal family for a proper rigid analytic space.

## INTRODUCTION

The purpose of the present paper is to formulate and develop a deformation theory of rigid analytic spaces. We will give a cohomological description of infinitesimal deformations. Furthermore we will prove an existence of a formal versal family for a proper rigid analytic space.

The original idea of deformations goes back to Kodaira and Spencer. They developed the theory of deformations of complex manifolds. Their deformation theory of complex manifolds is of great importance and becomes a standard tool for the study of complex analytic spaces. Our study of deformations of rigid analytic spaces is motivated by analogy to the case of complex analytic spaces. More precisely, our interest in the development of such a theory comes from two sources. First, we want to construct an analytic moduli theory via rigid analytic stacks by generalizing the classical deformation theory due to Kodaira-Spencer, Kuranishi and Grauert to the non-Archimedean theory. This viewpoint will be discussed in Section 5. Secondly, we may hope that our theory is useful in arithmetic geometry no less than the complex-analytic deformation theory is very useful in the study of complex-analytic spaces. Actually, our deformation theory will be one of the key ingredients of the generalization of the theory of  $p$ -adic period mappings due to Rapoport-Zink (cf. [20]).

Let  $K$  be a complete non-Archimedean valued field and let us consider deformations of affinoid  $K$ -algebras. Then we find a serious problem. As we see later (Remark 1.3), it occurs that local deformations of affinoid  $K$ -algebras as  $K$ -algebras are not necessary affinoid algebras. Thus contrary to the case of schemes, algebraic spaces, etc., we can not apply the theory of cotangent complexes of ringed topoi (cf. [17]) to our deformation theory. In this paper, we develop a general theory of cotangent complexes called *rigid cotangent complexes* which fits in with our deformations. By applying it, we will derive the following theorem.

**Theorem A.** *Let  $u : S \rightarrow S'$  be a closed immersion of rigid analytic spaces with nilpotent kernel  $\mathcal{I}$  of  $u^* : \mathcal{O}_{S'} \rightarrow \mathcal{O}_S$  such that  $\mathcal{I}^2 = 0$ . Let  $f : X \rightarrow S$  be a flat morphism of rigid analytic spaces over  $K$ .*

- (1) *The obstruction for the existence of the lifting of  $f$  to  $S'$  lies in  $\mathrm{Ext}_{\mathcal{O}_X}^2(\mathbb{L}_{X/S}^{\mathrm{rig}}, f^*\mathcal{I})$ .*
- (2) *If the obstruction  $o$  is zero, the set of isomorphism classes of the liftings of  $f$  to  $S'$  forms a torsor under  $\mathrm{Ext}_{\mathcal{O}_X}^1(\mathbb{L}_{X/S}^{\mathrm{rig}}, f^*\mathcal{I})$ .*
- (3) *Let  $\tilde{f} : \tilde{X} \rightarrow S'$  be a flat deformation of  $X$ . Then, the automorphism group of the lifting  $\tilde{X}$  is canonically isomorphic to  $\mathrm{Ext}_{\mathcal{O}_X}^0(\mathbb{L}_{X/S}^{\mathrm{rig}}, f^*\mathcal{I})$ .*

*The complex  $\mathbb{L}_{*/*}^{\mathrm{rig}}$  in the derived category  $\mathrm{D}^-(\mathcal{O}_X\text{-Mod})$  is a rigid cotangent complex defined in Section 2.*

The idea of rigid cotangent complexes comes from Raynaud's viewpoint of rigid geometry via formal schemes. Let  $X$  be a quasi-compact and quasi-separated rigid analytic space over  $K$ , and  $\mathcal{X}$  a formal  $R$ -scheme which is a formal model of  $X$ . Here  $R$  is the ring of integer of  $K$  (cf. [5]). Then the analytic differential module of  $X$  over  $K$  is described by the one of  $\mathcal{X}$  over  $R$  (cf. [7, 1.5]). If  $X$  is smooth, the analytic deformations of  $X$  is controlled by the analytic differential modules of  $X$ . Similarly, the rigid cotangent complex of  $X$  over  $K$  is constructed from the cotangent complex of the fixed formal model. Our theory is closely linked to formal-algebraic geometry and enable us to deal with deformations of rigid analytic spaces via formal schemes and the whole machinery of EGA (for example, flattening techniques by blowing-ups are very powerful). See Section 2 for the details. We remark that there is another approach to cotangent complex for rigid geometry due to O. Gabber-L. Ramero. It uses the theory of Huber's adic spaces rather than formal-algebraic geometry.

Our second goal is to show the existence of the formal versal deformation family which can be viewed as non-Archimedean formal analogue of Kuranishi and Grauert (cf. [19] [15]).

**Theorem B.** *Let  $X$  be a proper rigid analytic space over  $K$ . Then, there exists a formal versal deformation of  $X$ .*

In the appendix, we give a convenient criterion for an existence of square-zero deformations of a ringed topos. This criterion may be useful also in the other situations.

### Notations And Conventions

- (1) Unless otherwise stated,  $K$  will be a complete non-Archimedean valued field. We denote by  $K\langle X_1, \dots, X_n \rangle$  the Tate algebra in  $n$  indeterminates.
- (2) For the basic facts and definitions concerning rigid analytic spaces we refer to [3] [5] [6] [7].
- (3) For an affinoid algebra  $A$ , we denote by  $\mathrm{Sp}(A)$  the associated affinoid space (cf. [3, Chapter 7,8,9]).

- (4) All rigid analytic spaces in this paper will be quasi-compact and quasi-separated rigid analytic spaces. Quasi-separatedness means that the diagonal morphism  $X \rightarrow X \times X$  is quasi-compact.
- (5) *Properness* means Kiehl's properness (cf. [3]).
- (6) The *very weak topology* on an affinoid space means Grothendieck topology such that it has rational subdomains as the admissible open sets, and its admissible coverings are unions of finitely many rational subdomains. For general rigid analytic spaces, we define it in the same way. However, unless otherwise stated, we always equip rigid spaces with the *strong topology* in the sense of [3].
- (7) By  $|\bullet|_{sp}$  we mean the *spectral norm* (cf. [3, 3.2]).
- (8) Let  $\mathcal{A}$  be an admissible formal scheme (resp. an  $\mathcal{O}_{\mathcal{X}}$ -module on the admissible formal scheme  $\mathcal{X}$ ,...etc.) in the sense of [5]. Then we denote by  $\mathcal{A}^{\text{rig}}$  the rigid analytic spaces (resp. the  $\mathcal{O}_{\mathcal{X}^{\text{rig}}}$ -module on the rigid space  $\mathcal{X}^{\text{rig}}$ ,... etc.) defined as in [5, Section 4]. We say that  $\mathcal{A}$  is a formal model of  $\mathcal{A}^{\text{rig}}$ .
- (9) Let  $A$  be a ring and  $B$  an  $A$ -algebra. Let  $M$  be a  $B$ -module. A short exact sequence  $0 \rightarrow M \xrightarrow{i} E \xrightarrow{\pi} B \rightarrow 0$  where  $E$  is an  $A$ -algebra and  $\pi$  is a surjective homomorphism of  $A$ -algebras with  $i(M)^2 = 0$  in  $E$  is said to be a square-zero extension of  $B$  by  $M$  over  $A$ .

*Acknowledgement.* This paper forms a part of my master's thesis. I would like to thank Prof. Masao Aoki, Fumiharu Kato and Akio Tamagawa for helpful conversations and valuable comments concerning the issue presented in this paper, and my adviser Prof. Atsushi Moriwaki for continuous encouragement. I am supported by JSPS Fellowships for young scientists.

## 1. FIRST PROPERTIES OF LOCAL DEFORMATIONS

In this section, we will define and prove first properties of local deformations.

**Definition 1.1.** Let  $f : X \rightarrow S$  be a flat morphism (i.e., a flat morphism of the ringed sites) of rigid analytic spaces and  $S \rightarrow S'$  be a closed immersion of rigid analytic spaces with the nilpotent kernel  $\mathcal{I} := \text{Ker}(\mathcal{O}_{S'} \rightarrow \mathcal{O}_S)$ . We say that a pair  $(f' : X' \rightarrow S', \phi : X' \times_{S'} S \xrightarrow{\sim} X)$  is a deformation of  $f : X \rightarrow S$  to  $S'$  if the following properties are satisfied,

- (1)  $f'$  is a flat morphism of rigid analytic spaces,
- (2)  $\phi$  is an isomorphism of rigid analytic spaces.

A morphism from  $(f' : X' \rightarrow S', \phi : X' \times_{S'} S \xrightarrow{\sim} X)$  to  $(f'' : X'' \rightarrow S', \psi : X'' \times_{S'} S \xrightarrow{\sim} X)$  is a  $S'$ -morphism  $\alpha : X' \rightarrow X''$  of rigid analytic spaces such that  $\psi \circ \alpha|_S \circ \phi^{-1} = \text{id}_X$ .

**Example 1.2.** (1) Let  $S \rightarrow S'$  be a closed immersion of schemes locally of finite type over  $K$  with the nilpotent kernel  $\mathcal{I} := \text{Ker}(\mathcal{O}_{S'} \rightarrow \mathcal{O}_S)$ . Let  $f : X \rightarrow S$  be a flat morphism of  $K$ -schemes and suppose that  $X$  is of finite type over  $K$ . Let the pair  $(X'/S', X' \times_{S'} S \cong X)$  be a flat deformation of  $X$  to  $S'$ . Then the analytification  $(X'^{\text{an}}/S'^{\text{an}}, X'^{\text{an}} \times_{S'^{\text{an}}} S^{\text{an}} \xrightarrow{\sim} X^{\text{an}})$  is a flat deformation of  $X^{\text{an}}$  to  $S'^{\text{an}}$ . Here for any scheme  $W$  locally of finite type over  $K$ , we denote by  $W^{\text{an}}$  the associated rigid analytic space (See [2] for the analytifications).

(2) Let  $T$  be a split  $K$ -torus and  $M$  a split lattice of rank  $\dim(T)$  in the sense of [4]. Note that we can regard  $M$  as group  $K$ -scheme. If the closed immersion  $i : M \rightarrow T$  defines  $M$  a lattice of full rank in  $T$ , the quotient  $A := T/M$  is the rigid analytic group (See [4, p.661]). Let  $B$  be an Artin local  $K$ -algebra. As we see later (Lemma 4.1),  $B$  is an affinoid  $K$ -algebra. Let  $\tilde{i} : M \times_K B \rightarrow T \times_K B$  be a closed immersion which extends the morphism  $i$ . Then the rigid analytic group  $(T \times_K B)/(M \times_K B)$  over  $\text{Sp}(B)$  defines a deformation of  $A/K$  to  $B$ .

**Remark 1.3.** It may happen that a square-zero extension of an affinoid algebra is not an affinoid algebra. Let  $0 \rightarrow M \rightarrow E \rightarrow A \rightarrow 0$  be a square-zero extension of an affinoid algebra  $A$  over  $K$ , where  $M$  is a finitely generated  $A$ -module. By the fundamental theorem due to L. Illusie (cf. [17, Chapter 3, 1.2.3]), the set of isomorphism classes of square-zero extensions of  $A$  by  $M$  is classified by the group  $\text{Ext}_A^1(\mathbb{L}_{A/K}, M)$  where  $\mathbb{L}_{A/K}$  is the cotangent complex defined in [17, Chapter 2]. If  $E$  is an affinoid algebra and  $A$  is a Tate algebra  $K\langle T_1, \dots, T_r \rangle$ , then there exists a splitting  $A \rightarrow E$  of  $E \rightarrow A$  and hence, the extension class of this extension in  $\text{Ext}_A^1(\mathbb{L}_{A/K}, M)$  is zero. On the other hand, by Theorem A.1, there exists a canonical injective map  $\text{Ext}_A^1(\Omega_{A/K}^1, M) \rightarrow \text{Ext}_A^1(\mathbb{L}_{A/K}, M)$  where  $\Omega_{A/K}^1$  is the usual Kähler differential module. Therefore, in order to see the existence of such an extension with  $E$  not being an affinoid algebra, it suffices to show the following statement.

**Proposition 1.4.** *Let  $A = \mathbb{Q}_p\langle T \rangle$  be a Tate algebra over  $\mathbb{Q}_p$ . Then there exists a finitely generated  $A$ -module  $M$  such that  $\text{Ext}_A^1(\Omega_{A/\mathbb{Q}_p}^1, M)$  is non-zero.*

Proof. First we claim that, for any  $A$ -module  $M$ , we have  $\text{Ext}_A^1(\Omega_{A/\mathbb{Q}_p}^1, M) = \text{Ext}_A^1(\text{Ker}(\pi), M)$ , where  $\text{Ker}(\pi)$  is the kernel of the homomorphism

$$0 \rightarrow \text{Ker}(\pi) \rightarrow \Omega_{A/\mathbb{Q}_p}^1 \xrightarrow{\pi} \Omega_{A/\mathbb{Q}_p}^{\text{rig}} \rightarrow 0.$$

Here  $\Omega_{A/\mathbb{Q}_p}^{\text{rig}}$  is the differential module defined in [6, Section 1]. To prove our claim, we look at the associated long exact sequence to the short exact sequence

$$\text{Ext}_A^1(\Omega_{A/\mathbb{Q}_p}^{\text{rig}}, M) \rightarrow \text{Ext}_A^1(\Omega_{A/\mathbb{Q}_p}^1, M) \rightarrow \text{Ext}_A^1(\text{Ker}(\pi), M) \rightarrow \text{Ext}_A^2(\Omega_{A/\mathbb{Q}_p}^{\text{rig}}, M).$$

Then the claim follows easily from the fact that  $\Omega_{A/\mathbb{Q}_p}^{\text{rig}}$  is a free  $A$ -module.

Thus, what to prove is the existence of a finitely generated  $A$ -module  $M$  such that  $\text{Ext}_A^1(\text{Ker}(\pi), M)$  is non-zero. Suppose we have such an  $M$ , which is not necessarily finitely generated, then we can actually find a finitely generated  $A$ -module  $M$  having the same property. Indeed, Since  $A$  is a principal ideal domain, we have an  $A$ -injective resolution,  $0 \rightarrow A \rightarrow I \rightarrow I' \rightarrow 0$ . From the long exact sequence arising from this resolution, we see that  $\text{Ext}_A^2(\Omega_{A/\mathbb{Q}_p}^1, A) = 0$ . We have also an  $A$ -free resolution  $0 \rightarrow F' \rightarrow F \rightarrow N \rightarrow 0$  because every  $A$ -submodule of a free  $A$ -module is free. Then we derive  $\text{Ext}_A^1(\Omega_{A/\mathbb{Q}_p}^1, A) \neq 0$  from the long exact sequence,

$$\text{Ext}_A^1(\Omega_{A/\mathbb{Q}_p}^1, F') \rightarrow \text{Ext}_A^1(\Omega_{A/\mathbb{Q}_p}^1, F) \rightarrow \text{Ext}_A^1(\Omega_{A/\mathbb{Q}_p}^1, N) \rightarrow \text{Ext}_A^2(\Omega_{A/\mathbb{Q}_p}^1, F') \\ \text{and } \text{Ext}_A^2(\Omega_{A/\mathbb{Q}_p}^1, F') = 0.$$

The existence of a module  $M$  with  $\text{Ext}_A^1(\text{Ker}(\pi), M) \neq 0$  (not necessarily finitely generated) follows from the following lemma.

**Lemma 1.5.**  *$\text{Ker}(\pi)$  is a non-zero injective  $A$ -module. In particular  $\text{Ker}(\pi)$  is not  $A$ -projective.*

*Proof.* First, we will show  $\text{Ker}(\pi)$  is  $A$ -injective. To this aim, since  $A$  is a principal ideal domain, it suffices to prove that  $\text{Ker}(\pi)$  is a divisible  $A$ -module. Let  $m$  and  $a$  be elements in  $\text{Ker}(\pi)$  and  $A$  respectively. We want to find an element  $n$  of  $\text{Ker}(\pi)$  such that  $m = an$ . By induction, we may assume that  $a$  is irreducible. However there exists the exact sequence

$$0 \rightarrow \text{Hom}(\Omega_{A/\mathbb{Q}_p}^{\text{rig}}, A/(a)) \xrightarrow{\xi} \text{Hom}(\Omega_{A/\mathbb{Q}_p}^1, A/(a)) \rightarrow \text{Hom}(\text{Ker}(\pi), A/(a)) \rightarrow 0$$

where  $\xi$  is an isomorphism by  $\text{Hom}(\Omega_{A/\mathbb{Q}_p}^{\text{rig}}, A/(a)) \cong \text{Der}_{\mathbb{Q}_p}(A, A/(a)) \cong \text{Hom}(\Omega_{A/K}^1, A/(a))$ . Thus  $\text{Ker}(\pi) \otimes_A A/(a) = 0$ . Finally  $d(\exp(pT))$  lies in  $\text{Ker}(\pi)$  because  $\exp(pT) = \sum_{n=0}^{\infty} (pT)^n/n!$  is transcendental over  $\mathbb{Q}_p(T)$ .  $\square$

Next we investigate the stability of the Grothendieck topology of a rigid analytic space under deformations. First of all, we will consider a class of morphisms of rigid analytic spaces defined as follows. A morphism  $f : X \rightarrow Y$  is said to be of *affinoid type* if  $f$  is the composite map  $X \xrightarrow{g} Y \times_K \text{Sp}(K\langle T_1, \dots, T_r \rangle) \xrightarrow{pr_1} Y$  for some positive integer  $r$  and some closed immersion  $g$ .

**Proposition 1.6.** *Let  $u : S \rightarrow S'$  be a closed immersion of rigid analytic spaces with nilpotent kernel  $\text{Ker}(u^* : \mathcal{O}_{S'} \rightarrow \mathcal{O}_S)$  and  $X$  a rigid analytic space. Suppose  $f : X \rightarrow S$  be a morphism of affinoid type and  $f' : \tilde{X} \rightarrow S'$  is a (possibly non-flat) deformation of  $f$ . Then,  $f'$  is a morphism of affinoid type.*

*Proof.* First we prove the case when  $S := \text{Sp}(A)$ ,  $S' := \text{Sp}(A')$  and  $I := \text{Ker}(u^* : A' \rightarrow A)$ . Suppose that  $I^n = 0$ . By induction on  $n$ , it suffices

to consider the case where  $I^2 = 0$ . Now we have an exact sequence

$$0 \longrightarrow I\mathcal{O}_{\tilde{X}} \longrightarrow \mathcal{O}_{\tilde{X}} \longrightarrow \mathcal{O}_X \longrightarrow 0.$$

Since  $I^2 = 0$ ,  $I\mathcal{O}_{\tilde{X}} = I\mathcal{O}_X$ . Thus by Tate's acyclic theorem ([3, 8.2]), we have

$$\check{H}^1(\tilde{\mathcal{U}}, I\mathcal{O}_{\tilde{X}}) = \check{H}^1(\mathcal{U}, I\mathcal{O}_X) = (0)$$

where  $\tilde{\mathcal{U}} = \{\tilde{U}_i\}_{i=0}^r$  is a finite affinoid cover on  $\tilde{X}$  and  $\mathcal{U} = \{U_i\}_{i=0}^r$  is the reduction of  $\tilde{\mathcal{U}}$  to  $X$ . By  $\check{H}(\tilde{\mathcal{U}}, \bullet)$ , we mean the Čech cohomology with respect to  $\tilde{\mathcal{U}} = \{\tilde{U}_i\}$ . By considering the long exact sequence of the Čech cohomology with respect to the very weak topology on  $\tilde{X}$ , we have an exact sequence

$$0 \longrightarrow H^0(X, I) \longrightarrow H^0(\tilde{X}, \mathcal{O}_{\tilde{X}}) \longrightarrow H^0(X, \mathcal{O}_X) \longrightarrow 0.$$

We put  $H^0(X, \mathcal{O}_X) = A\langle X_1, \dots, X_n \rangle / J$ . Let us construct a surjective map from  $A'\langle T_1, \dots, T_r \rangle$  to  $H^0(\tilde{X}, \mathcal{O}_{\tilde{X}})$  which extends a natural surjective map  $A\langle X_1, \dots, X_n \rangle \rightarrow H^0(X, \mathcal{O}_X)$ . Let  $\xi_1, \dots, \xi_n$  be elements of  $H^0(\tilde{X}, \mathcal{O}_{\tilde{X}})$  which are liftings of  $X_1, \dots, X_n$ , respectively. Then, we have the following diagram

$$\begin{array}{ccc} A'[X_1, \dots, X_n] & \longrightarrow & A\langle X_1, \dots, X_n \rangle \\ \downarrow \pi & & \downarrow \\ H^0(\tilde{X}, \mathcal{O}_{\tilde{X}}) & \longrightarrow & H^0(X, \mathcal{O}_X), \end{array}$$

where  $\pi$  is defined by  $X_i \rightarrow \xi_i$ . The right vertical arrow and the lower horizontal arrow are surjective. We want to see that  $\pi$  is continuous when we equip  $K[X_1, \dots, X_n]$  with the Gaussian norm. If necessary, we can choose  $\xi_1, \dots, \xi_n$  such that  $\max_{1 \leq k \leq r} |\text{res}_{\tilde{U}_k}(\xi_i)|_{sp} \leq 1$  for  $1 \leq i \leq n$  where  $\text{res}_{\tilde{U}_k}(\xi_i)$  is the restriction of  $\xi_i$  to  $H^0(\tilde{U}_k, \mathcal{O}_{\tilde{X}})$ . Indeed, by [3, 3.8.2.2],  $|X_i|_{sp} \leq |X_i| \leq 1$  on  $A\langle X_1, \dots, X_n \rangle / J$  and thus we have the same inequality on  $U_i$  for  $1 \leq i \leq r$  by [3, 3.8.1.4]. Hence, we have  $|\text{res}_{\tilde{U}_k}(\xi_i)|_{sp} \leq 1$  for any  $i, k$ . This implies the composite map

$$A'[X_1, \dots, X_n] \xrightarrow{\pi} H^0(\tilde{X}, \mathcal{O}_{\tilde{X}}) \rightarrow \bigoplus_{i=0}^r H^0(\tilde{U}_i, \mathcal{O}_{\tilde{U}_i})$$

is a continuous map. Now by Tate's acyclic theorem we have

$$H^0(\tilde{X}, \mathcal{O}_{\tilde{X}}) = \text{Ker}(\bigoplus_{i=0}^r H^0(\tilde{U}_i, \mathcal{O}_{\tilde{U}_i}) \rightrightarrows \bigoplus_{i < j} H^0(\tilde{U}_i \cap \tilde{U}_j, \mathcal{O}_{\tilde{U}_i \cap \tilde{U}_j})).$$

Note that the topology of  $\bigoplus_{i=0}^r H^0(\tilde{U}_i, \mathcal{O}_{\tilde{U}_i})$  (i.e. direct sum of the topologies on affinoid algebras  $H^0(\tilde{U}_i, \mathcal{O}_{\tilde{U}_i})$ ) induces the topology on  $H^0(\tilde{X}, \mathcal{O}_{\tilde{X}})$  which is complete. Thus we see  $\pi$  is also continuous. From this, there exists a unique homomorphism

$$\Pi : A'\langle X_1, \dots, X_n \rangle \longrightarrow H^0(\tilde{X}, \mathcal{O}_{\tilde{X}})$$

which extends the homomorphism  $\pi$ . We claim that  $\Pi$  is surjective. Indeed, the kernel of the surjection  $H^0(\tilde{X}, \mathcal{O}_{\tilde{X}}) \rightarrow H^0(X, \mathcal{O}_X)$  is  $IH^0(\tilde{X}, \mathcal{O}_{\tilde{X}})$ . On the other hand, it is clear that the image of  $\pi$  contains  $IH^0(\tilde{X}, \mathcal{O}_{\tilde{X}})$ . Hence,

we see that  $\Pi$  is a surjection. Therefore, there exists the commutative diagram of rigid analytic spaces

$$\begin{array}{ccc} X & \xrightarrow{\quad i \quad} & \tilde{X} \\ \alpha \downarrow & & \downarrow \beta \\ \mathbb{D}_A^n & \xrightarrow[\quad j \quad]{} & \mathbb{D}_{A'}^n \end{array}$$

where  $\mathbb{D}_A^n$  and  $\mathbb{D}_{A'}^n$  are  $\mathrm{Sp}(A\langle X_1, \dots, X_n \rangle)$  and  $\mathrm{Sp}(A'\langle X_1, \dots, X_n \rangle)$  respectively.  $i$  and  $j$  are maps which induce bijective maps of underlying sets. The morphism  $j \circ \alpha$  induce an injective map of underlying sets. Hence  $\beta$  is bijective onto the image of  $j \circ \alpha$  as sets. To prove the proposition, it suffices to show that homomorphism  $\mathcal{O}_{\mathbb{D}_{A'}, \beta(x)} \xrightarrow{\beta^*} \mathcal{O}_{\tilde{X}, x}$  is surjection for any point  $x$  of  $\tilde{X}$ . Let  $a$  be an element of  $\mathcal{O}_{\tilde{X}, x}$ . There is an element  $b$  in  $\mathcal{O}_{\mathbb{D}_A^n, \beta(x)}$  such that  $\alpha^*(b) = i^*(a)$ . Moreover there is an element  $c$  in  $\mathcal{O}_{\mathbb{D}_{A'}, \beta(x)}$  such that  $j^*(c) = b$ . Then,  $\beta^*(c) - a$  is in  $\mathrm{Ker}(i^*) = I\mathcal{O}_{\tilde{X}, x}$ . Since  $I^2 = 0$ , we have  $I \mathrm{Image}(\beta^*) = I\mathcal{O}_{\tilde{X}, x}$ . Thus we have an element  $d$  in  $\mathcal{O}_{\mathbb{D}_{A'}, \beta(x)}$  such that  $\beta^*(d + c) = a$ , since  $I$  is contained by  $\mathrm{Image}(\beta^*)$ .

Finally the assertion for the general case follows from the proof of the local case.  $\square$

**Proposition 1.7.** *Let  $u : A' \rightarrow A$  be a surjective homomorphism of affinoid  $K$ -algebras with the nilpotent kernel and  $X$  be a  $K$ -affinoid space over  $\mathrm{Sp}(A)$ . Let  $U$  be a rational subdomain of  $X$ . Suppose  $\tilde{X} \rightarrow \mathrm{Sp}(A')$  is a (possibly non-flat) deformation of  $X$  and  $\tilde{X}$  is an affinoid spaces. Then, the lifting  $\tilde{U}$  of  $U$  in  $\tilde{X}$  is a rational subdomain of  $\tilde{X}$ .*

*Proof.* We put

$$\begin{aligned} U &= X(f_0, \dots, f_r) \\ &= \{x \in X \mid |f_1(x)| \leq |f_0(x)|, \dots, |f_r(x)| \leq |f_0(x)|\} \\ &= \mathrm{Sp}(H^0(X, \mathcal{O}_X)\langle T_1, \dots, T_r \rangle / (f_1 - T_1 f_0, \dots, f_r - T_r f_0)) \end{aligned}$$

where  $f_0, \dots, f_r$  generates the unit ideal. We choose elements  $\tilde{f}_0, \dots, \tilde{f}_r \in H^0(\tilde{X}, \mathcal{O}_{\tilde{X}})$  which are the liftings of  $f_0, \dots, f_r$  respectively. Since  $X \rightarrow \tilde{X}$  is a nilpotent thickening, we have

$$\begin{aligned} \tilde{U} &= \tilde{X}(\tilde{f}_0, \dots, \tilde{f}_r) \\ &= \{x \in \tilde{X} \mid |\tilde{f}_1(x)| \leq |\tilde{f}_0(x)|, \dots, |\tilde{f}_r(x)| \leq |\tilde{f}_0(x)|\} \\ &= \mathrm{Sp}(H^0(\tilde{X}, \mathcal{O}_{\tilde{X}})\langle T_1, \dots, T_r \rangle / (\tilde{f}_1 - T_1 \tilde{f}_0, \dots, \tilde{f}_r - T_r \tilde{f}_0)). \end{aligned}$$

This implies the proposition.  $\square$

**Theorem 1.8.** *Let  $u : S \rightarrow S'$  be a closed immersion of rigid analytic spaces with nilpotent kernel  $\mathrm{Ker}(u^* : \mathcal{O}_{S'} \rightarrow \mathcal{O}_S)$  and  $f : X \rightarrow S$  a morphism of*

*rigid analytic spaces. Let  $f' : \tilde{X} \rightarrow S'$  be a (possibly non-flat) deformation of  $f$ . Suppose  $U$  is an admissible open set with respect to the strong topology on  $X$ . Then, the lifting  $\tilde{U}$  of  $U$  in  $\tilde{X}$  is the admissible open set of  $\tilde{X}$ . Similarly, an admissible covering of  $X$  lifts to admissible covering of  $\tilde{X}$ . In particular the Grothendieck topology of a rigid analytic space is stable under nilpotent deformations.*

*Proof.* By Proposition 1.6 and [3, 9.1.2.2], we may assume that  $X, \tilde{X}, S$  and  $S'$  are affinoid spaces. Next, we note that the strongest topology among the topologies which are slightly finer (See for the definition [3, 9.1.2.1]) than very weak topology coincides with the strong topology by the theorem of Gerritzen-Grauert [3, 7.3.5.3].

Thus to prove the assertion, it suffices to check that (cf. [3, 9.1.4.2]):

- (1) The set  $\tilde{U}$  admits a covering  $\{\tilde{U}_i\}_i$  by affinoid subdomains  $\tilde{U}_i \subset \tilde{X}$  such that, for any affinoid morphism  $\phi : Y \rightarrow \tilde{X}$  with  $\phi(Y) \subset \tilde{U}$ , the covering  $\{\phi^{-1}(\tilde{U}_i)\}_i$  of  $Y$  has a finite rational subdomain covering which refines it.
- (2) Let  $\{\tilde{V}_j\}_j$  be a covering which are the liftings of an admissible covering of the admissible open set  $V \subset X$ . Note that if the first half of our claim is verified,  $V_i$  are admissible. Let  $\tilde{V} \subset \tilde{X}$  be the lifting of  $V$ . Then, for any affinoid morphism  $\phi : Y \rightarrow \tilde{X}$  with  $\phi(Y) \subset \tilde{V}$ , the covering  $\{\phi^{-1}(\tilde{V}_i)\}_i$  of  $Y$  has a finite rational subdomain covering which refines it.

Since  $|X| = |\tilde{X}|$ , we can easily check these conditions by using Proposition 1.7.  $\square$

## 2. RIGID COTANGENT COMPLEX

Since the theory of cotangent complex for ringed topoi (cf. [17]) can not be applied to our problem, we define a *rigid cotangent complex* inspired by [7, Proposition 1.5].

**2.1. Analytic cotangent complex of formal schemes.** First of all, following [13], we will recall the analytic cotangent complexes for formal schemes locally of topologically finite presentation over  $\mathrm{Spf} R$ , where  $R$  is the ring of integer of  $K$ . Let  $\pi$  be an element in the maximal ideal of  $R$  and  $A$  a complete  $R$ -algebra topologically of finite presentation. Consider the  $\pi$ -adic completion functor

$$(A\text{-Mod}) \longrightarrow (A\text{-Mod}) \\ L \mapsto L^\wedge.$$

This functor induces the derived functor

$$D^-(A\text{-Mod}) \longrightarrow D^-(A\text{-Mod}).$$

Indeed, for any quasi-isomorphism of complexes of flat  $A$ -modules  $K^\bullet \rightarrow L^\bullet$ , the induced homomorphism  $(K^\bullet)^\wedge \rightarrow (L^\bullet)^\wedge$  ((complex) $^\wedge$  denotes the



termwise completion) is a quasi-isomorphism (cf. [13, 7.1.11]). Moreover, the derived category  $D^-(R\text{-Mod})$  is naturally identified with the localization of the homotopy category  $K^-(A\text{-flat Mod})$  up to quasi-isomorphisms. Thus  $\pi$ -adic completion functor induces the derived functor.

Let  $\phi : A \rightarrow B$  be a homomorphism of complete  $R$ -algebras of topologically finite presentation. The  $B$ -module of analytic differentials relative to  $\phi$  is defined as  $\Omega_{B/A}^{\text{an}} := \Omega_{B/A}^{1\wedge}$ . The analytic cotangent complex of  $\phi$  is the complex  $\mathbb{L}_{B/A}^{\text{an}} := \mathbb{L}_{B/A}^\wedge$ . Here  $\mathbb{L}_{B/A}$  is the usual cotangent complex of  $\phi$  (cf. [17]). There exists a natural map  $\mathbb{L}_{B/A}^{\text{an}} \rightarrow \Omega_{B/A}^{\text{an}}$  and an isomorphism  $H_0(\mathbb{L}_{B/A}^{\text{an}}) \cong \Omega_{B/A}^{\text{an}}$ . If  $\phi$  is smooth, then there is a natural quasi-isomorphism  $\mathbb{L}_{B/A}^{\text{an}} \cong \Omega_{B/A}^{\text{an}}[0]$ .

Next we define the analytic cotangent complex for formal schemes locally of topologically finite presentation over  $\text{Spf } R$  by gluing the complexes constructed as above. Let  $f : \mathcal{X} \rightarrow \mathcal{Y}$  be a morphism of formal schemes locally of topologically finite presentation over  $\text{Spf } R$  (cf. [5]). We suppose that  $\mathcal{Y}$  is affine for a while. For an affine open set  $\mathcal{U}$  in  $\mathcal{X}$ , the small category  $F_{\mathcal{U}}$  of all affine open sets  $\mathcal{V}$  in  $\mathcal{Y}$  with  $f(\mathcal{U}) \subset \mathcal{V}$  is a cofiltered family under inclusion. For every  $\mathcal{V} \in F_{\mathcal{U}}$ ,  $\mathcal{O}_{\mathcal{Y}}(\mathcal{V})$  is a complete  $R$ -algebra of topologically finite presentation. The cofiltered family of maps  $\mathcal{O}_{\mathcal{Y}}(\mathcal{V}) \rightarrow \mathcal{O}_{\mathcal{X}}(\mathcal{U})$  gives rise to the correspondence

$$\mathcal{U} \longrightarrow \mathbb{L}(\mathcal{U}/\mathcal{Y}) := \text{colim}_{\mathcal{V} \in F_{\mathcal{U}}} \mathbb{L}_{\mathcal{O}_{\mathcal{X}}(\mathcal{U})/\mathcal{O}_{\mathcal{Y}}(\mathcal{V})}^{\text{an}}.$$

Now note that every usual cotangent complex is constructed as a complex of free modules in a functorial fashion. Hence for any homomorphism  $A \rightarrow B$ , we can construct the analytic cotangent complex of  $A \rightarrow B$  as a complex of flat  $B$ -modules in a functorial fashion. (The flatness follows from the fact that for any  $B$ -flat module  $F$ , the completion  $F^\wedge$  is also  $B$ -flat (cf. for example [13, 7.1.6 (1)])). Therefore, by [10, Chapter 0, 3.2.1], we can extend the above correspondence to the complex of presheaves on  $\mathcal{X}$ . The analytic cotangent complex is defined as the complex of termwise the associated sheaves. Finally for a general formal scheme  $\mathcal{Y}$ , we can construct the complex by gluing the complex constructed locally on  $\mathcal{Y}$ .

The following proposition will be used in the next subsection.

**Proposition 2.1.** *Let  $f : \mathcal{X} \rightarrow \mathcal{Y}$  and  $g : \mathcal{Z} \rightarrow \mathcal{Y}$  be morphisms of formal schemes locally of topologically finite presentation over  $\text{Spf } R$ . Let  $\mathcal{X} \times_{\mathcal{Y}} \mathcal{Z}$  be the fibre product of  $\mathcal{X}$  and  $\mathcal{Z}$  over  $\mathcal{Y}$  in the category of formal  $R$ -schemes. Then there exists a natural quasi-isomorphism*

$$L \text{pr}_1^* \mathbb{L}_{\mathcal{X}/\mathcal{Y}}^{\text{an}} \xrightarrow{\sim} \mathbb{L}_{\mathcal{X} \times_{\mathcal{Y}} \mathcal{Z}/\mathcal{Z}}^{\text{an}}$$

where  $\text{pr}_1 : \mathcal{X} \times_{\mathcal{Y}} \mathcal{Z} \rightarrow \mathcal{X}$  is the first projection.

*Proof.* Since our assertion is a local issue, we may suppose that  $\mathcal{X}$ ,  $\mathcal{Y}$  and  $\mathcal{Z}$  are affine. Set  $\mathcal{X} = \text{Spf } B$ ,  $\mathcal{Y} = \text{Spf } A$  and  $\mathcal{Z} = \text{Spf } C$ . First we shall show that there exists a natural quasi-isomorphism  $\phi : (B \hat{\otimes}_A C) \otimes_B \mathbb{L}_{B/A}^\wedge \rightarrow$

$\mathbb{L}_{(B\hat{\otimes}_A C)/C}^\wedge$ . We remark that  $\mathrm{Spf} B\hat{\otimes}_A C$  is the fibre product of  $\mathrm{Spf} B$  and  $\mathrm{Spf} C$  over  $\mathrm{Spf} A$  in the category of formal  $R$ -schemes. In this proof, unless otherwise stated, we view complexes as just complexes (not objects in the derived category) and we denote by  $\xrightarrow{\mathrm{qis}}$  a quasi-isomorphism of complexes. Actually, usual cotangent complexes are constructed as complexes of flat modules via standard resolutions and their completions are also flat. There exists natural quasi-morphisms  $((B\hat{\otimes}_A C) \otimes_{(B\otimes_A C)} (B \otimes_A C) \otimes_B \mathbb{L}_{B/A})^\wedge \xrightarrow{\sim} ((B\hat{\otimes}_A C) \otimes_B \mathbb{L}_{B/A})^\wedge \xrightarrow{\mathrm{qis}} ((B\hat{\otimes}_A C) \otimes_B \mathbb{L}_{B/A}^\wedge)^\wedge$ . The second quasi-isomorphism follows from [13, Lemma 7.1.22]. On the other hand, by the base change theorem of usual cotangent complexes [17, Chapter 2, 2.2], we have a natural quasi-isomorphism  $((B\hat{\otimes}_A C) \otimes_{(B\otimes_A C)} (B \otimes_A C) \otimes_B \mathbb{L}_{B/A})^\wedge \xrightarrow{\mathrm{qis}} ((B\hat{\otimes}_A C) \otimes_{(B\otimes_A C)} \mathbb{L}_{(B\otimes_A C)/C})^\wedge$ . The next claim implies a quasi-isomorphism  $(B\hat{\otimes}_A C) \otimes_B \mathbb{L}_{B/A}^\wedge \xrightarrow{\mathrm{qis}} ((B\hat{\otimes}_A C) \otimes_B \mathbb{L}_{B/A}^\wedge)^\wedge$  and thus we have a natural quasi-isomorphism  $(B\hat{\otimes}_A C) \otimes_B \mathbb{L}_{B/A}^\wedge \xrightarrow{\sim} ((B\hat{\otimes}_A C) \otimes_{(B\otimes_A C)} \mathbb{L}_{(B\otimes_A C)/C})^\wedge$  in the derived category.

**Claim 2.1.1.** *Let  $A \rightarrow B$  be a homomorphism of admissible  $R$ -algebras and  $K^\bullet := \mathbb{L}_{B/A}^{\mathrm{an}}$  the analytic cotangent complex. Then there is a natural quasi-isomorphism*

$$K^\bullet \otimes_A^\mathbb{L} B \xrightarrow{\sim} (K^\bullet \otimes_A^\mathbb{L} B)^\wedge$$

in the derived category  $\mathrm{D}^-(A\text{-Mod})$ .

*Proof.* It suffices to show that for any positive integer  $n$ , the truncation

$$\tau_{[-n]}(K^\bullet \otimes_A^\mathbb{L} B) \xrightarrow{\sim} \tau_{[-n]}(K^\bullet \otimes_A^\mathbb{L} B)^\wedge$$

is a quasi-isomorphism. Due to [13, 7.1.15, 7.1.33 (1)] and the fact that  $B$  is coherent, we can assume that  $\tau_{[-n]}K^\bullet$  is a complex of free  $A$ -modules of finite type. Therefore it suffices to show that for every free  $A$ -module of finite type  $F$ , the natural homomorphism  $F \otimes_A B \rightarrow (F \otimes_A B)^\wedge$  is an isomorphism. However it is clear.  $\square$

Therefore, to see that  $\phi$  is a quasi-isomorphism, it suffices to show that the natural morphism  $\psi : ((B\hat{\otimes}_A C) \otimes_{B\otimes_A C} \mathbb{L}_{(B\otimes_A C)/C})^\wedge \rightarrow \mathbb{L}_{(B\hat{\otimes}_A C)/C}^\wedge$  is a quasi-isomorphism. Note that, by the base-change theorem of usual cotangent complexes [17, Chapter 2, 2.2],  $((B\hat{\otimes}_A C) \otimes_{B\otimes_A C} \mathbb{L}_{(B\otimes_A C)/C})_n^\wedge := ((B\hat{\otimes}_A C) \otimes_{B\otimes_A C} \mathbb{L}_{(B\otimes_A C)/C})^\wedge \otimes_R (R/\pi^n R) \xrightarrow{\mathrm{qis}} \mathbb{L}_{(B_n \otimes_{A_n} C_n)/C_n}^\wedge$ ,  $(\mathbb{L}_{(B\hat{\otimes}_A C)/C}^\wedge)_n := \mathbb{L}_{(B\hat{\otimes}_A C)/C}^\wedge \otimes_R (R/\pi^n R) \xrightarrow{\mathrm{qis}} \mathbb{L}_{(B_n \otimes_{A_n} C_n)/C_n}^\wedge$  where  $B_n = B \otimes_R (R/\pi^n R)$ ,  $A_n = A \otimes_R (R/\pi^n R)$  and  $C_n = C \otimes_R (R/\pi^n R)$ . Thus the reduction  $\psi \otimes_R (R/\pi^n R)$  is a quasi-isomorphism. Now consider the right derived functor

$$R\lim : \mathrm{D}((R\text{-Mod})^{\mathbb{N}}) \rightarrow \mathrm{D}(R\text{-Mod})$$

where  $(R\text{-Mod})^{\mathbb{N}}$  is the projective system of  $R$ -modules and  $\lim : (R\text{-Mod})^{\oplus \mathbb{N}} \rightarrow R\text{-Mod}$ ,  $\{M_i, d_i : M_i \rightarrow M_{i-1}\}_{i \geq 1} \mapsto \text{proj. lim}_i M_i$  is the inverse limit functor. The projective systems of complexes  $((B \hat{\otimes}_A C) \otimes_{B \otimes_A C} \mathbb{L}_{(B \otimes_A C)/C}^\wedge)_n)_{n \geq 1}$  and  $((\mathbb{L}_{(B \hat{\otimes}_A C)/C}^\wedge)_n)_{n \geq 1}$  are acyclic for the functor  $\lim$  because they are consisting of surjections. Thus we have that

$$R \lim((B \hat{\otimes}_A C) \otimes_{B \otimes_A C} \mathbb{L}_{(B \otimes_A C)/C}^\wedge)_n)_{n \geq 1} = ((B \hat{\otimes}_A C) \otimes_{B \otimes_A C} \mathbb{L}_{(B \otimes_A C)/C}^\wedge)^\wedge$$

and

$$R \lim((\mathbb{L}_{(B \hat{\otimes}_A C)/C}^\wedge)_n)_{n \geq 1} = \mathbb{L}_{(B \hat{\otimes}_A C)/C}^\wedge.$$

Since there exists a quasi-isomorphism  $R \lim((B \hat{\otimes}_A C) \otimes_{B \otimes_A C} \mathbb{L}_{(B \otimes_A C)/C}^\wedge)_n)_{n \geq 1} \xrightarrow{\text{qis}} R \lim((\mathbb{L}_{(B \hat{\otimes}_A C)/C}^\wedge)_n)_{n \geq 1}$ , we see that  $\psi$  is a quasi-isomorphism and thus  $\phi$  is

a quasi-isomorphism. Furthermore, a natural homomorphism  $(B \hat{\otimes}_A C) \otimes_B \mathbb{L}_{B/A}^\wedge \rightarrow \mathbb{L}_{(B \hat{\otimes}_A C)/C}^\wedge$  is a quasi-isomorphism (in the derived category) because  $\mathbb{L}_{B/A}^\wedge$  consists of flat  $B$ -modules.

Next note that  $(B \hat{\otimes}_A C) \otimes_B \mathbb{L}_{B/A}^\wedge$  and  $\mathbb{L}_{(B \hat{\otimes}_A C)/C}^\wedge$  are pseudo-coherent<sup>1</sup>. Let  $L_i^\Delta$  (resp.  $M_i^\Delta$ ) be the sheaf of the coherent  $\mathcal{O}_{\text{Spf}(B \hat{\otimes}_A C)}$ -module associated to the coherent  $B \hat{\otimes}_A C$ -module  $L_i := H_i((B \hat{\otimes}_A C) \otimes_B \mathbb{L}_{B/A}^\wedge)$  (resp.  $M_i := H_i(\mathbb{L}_{(B \hat{\otimes}_A C)/C}^\wedge)$ ). To complete the proof of the proposition, it suffices to show that following claim.

**Claim 2.1.2.** *Under the same assumption as above, there exists natural isomorphisms*

$$L_i^\Delta \cong H_i(L \text{pr}_1^* \mathbb{L}_{\text{Spf } B/\text{Spf } A}^{\text{an}})$$

and

$$M_i^\Delta \cong H_i(\mathbb{L}_{(\text{Spf } B \hat{\otimes}_A C)/\text{Spf } C}^{\text{an}}).$$

*Proof.* The second assertion can be shown by the same way of the proof of the first assertion. Hence we will prove the first quasi-isomorphism. What to prove is that the natural isomorphism  $\xi : L_i^\Delta \rightarrow H_i(L \text{pr}_1^* \mathbb{L}_{\text{Spf } B/\text{Spf } A}^{\text{an}})$  induces an isomorphism on each stalk. Let  $U := \text{Spf } R(U)$ ,  $V := \text{Spf } R(V)$ , and  $W := \text{Spf } R(W)$  be affine open sets of  $\text{Spf } B \hat{\otimes}_A C$ ,  $\text{Spf } B$ , and  $\text{Spf } A$  respectively and suppose that  $\text{pr}_1(U) \subset V$  and  $f(V) \subset W$ . By the transitivity of analytic cotangent complexes [13, 7.1.33 (2)], we easily see that there exists a natural isomorphism  $H_i(R(U) \otimes_{R(V)} \mathbb{L}_{R(V)/R(W)}^\wedge) \cong H_i((R(U) \otimes_B$

<sup>1</sup>Let  $n$  be an integer. We say that a complex of  $R$ -module  $K^\bullet$  is  $n$ -pseudo-coherent if there exists a quasi isomorphism  $C^\bullet \rightarrow K^\bullet$  where  $C^\bullet$  is a complex bounded above and  $C^k$  is a finitely generated free  $R$ -module for every  $k \geq n$ . We say that  $K^\bullet$  is pseudo-coherent if  $K^\bullet$  is  $n$ -pseudo-coherent for every integer  $n$ .  $n$ -pseudo-coherence is stable under quasi-isomorphisms.

$\mathbb{L}_{B/A}^\wedge$ ). Therefore we have natural isomorphisms

$$\begin{aligned}
H_i(L \operatorname{pr}_1^* \mathbb{L}_{\operatorname{Spf} B / \operatorname{Spf} A}^{\operatorname{an}})_x &\cong \operatorname{colim}_{\substack{x \in U \\ \text{affine open}}} \operatorname{colim}_{\substack{\operatorname{pr}_1(U) \subset V \\ \text{affine open}}} \operatorname{colim}_{\substack{f(V) \subset W \\ \text{affine open}}} H_i(R(U) \otimes_{R(V)} \mathbb{L}_{R(V)/R(W)}^\wedge) \\
&\cong \operatorname{colim}_{\substack{x \in U \\ \text{affine open}}} H_i(R(U) \otimes_B \mathbb{L}_{B/A}^\wedge) \\
&\cong \operatorname{colim}_{\substack{x \in D(c) \\ c \in B \hat{\otimes}_A C}} H_i(R(D(c)) \otimes_B \mathbb{L}_{B/A}^\wedge) \\
&\cong L_i \otimes_{(B \hat{\otimes}_A C)} \mathcal{O}_{\operatorname{Spf}(B \hat{\otimes}_A C), x}
\end{aligned}$$

where  $D(c) = \{x \in \operatorname{Spf}(B \hat{\otimes}_A C), c \notin \mathfrak{m}_x\}$  and  $R(D(c)) := H^0(D(c), \mathcal{O}_{\operatorname{Spf}(B \hat{\otimes}_A C)})$ . The final isomorphism follows from the next lemma.

**Lemma 2.2.** *Let  $A$  be a complete  $R$ -algebra of topologically finite presentation. For any  $c \in A$ , let  $D(c) := \{x \in \operatorname{Spf} A; c \notin \mathfrak{m}_x\}$ . Then the natural map  $A \rightarrow H^0(D(c), \mathcal{O}_{\operatorname{Spf} A})$  is flat.*

*Proof.* Note that  $H^0(D(c), \mathcal{O}_{\operatorname{Spf} A})$  is the  $\pi$ -adic completion of  $A_c$ . Then our claim follows from the fact that the  $\pi$ -adic completion of every flat  $A$ -module is also  $A$ -flat.  $\square$

**2.2. Rigid cotangent complex.** In the rest of this section we often use the terminologies in [5] [6] [7]. For the definition of admissible formal schemes, admissible blow-up, smoothness, etc., we refer to [5] [6] [7]. Before we define rigid cotangent complexes, we will show some results which are used later.

**Lemma 2.3.** (1) *Let  $X \xrightarrow{f} Y \xrightarrow{g} Z$  be a sequence of morphisms of rigid analytic spaces. Then there is a natural quasi-isomorphism*

$$L(g \circ f)^* \xrightarrow{\sim} Lf^* \circ Lg^*.$$

(2) *Let  $\mathcal{X} \xrightarrow{\tilde{f}} \mathcal{Y} \xrightarrow{\tilde{g}} \mathcal{Z}$  be a sequence of admissible formal  $R$ -schemes which is a formal model of  $X \xrightarrow{f} Y \xrightarrow{g} Z$  (Actually we can choose such a sequence of formal schemes for any sequence of rigid analytic spaces by [5, Theorem 4.1]). Then there is a natural quasi-isomorphism*

$$(L\tilde{f}^*(\mathbb{L}_{\mathcal{Y}/\mathcal{Z}}^{\operatorname{an}}))^{\operatorname{rig}} \xrightarrow{\sim} Lf^*(\mathbb{L}_{Y/Z}^{\operatorname{an}})^{\operatorname{rig}}.$$

*Proof.* (1) Our claim follows from Grothendieck spectral sequence (cf. [18, 1.8.7]).

(2) First, note that the assertion is local on  $\mathcal{X}$ . Thus we suppose that  $\mathcal{X}$  and  $\mathcal{Y}$  are affine. Put  $A = H^0(\mathcal{X}, \mathcal{O}_{\mathcal{X}})$  and  $B = H^0(\mathcal{Y}, \mathcal{O}_{\mathcal{Y}})$ . Moreover, we may replace  $\mathbb{L}_{\mathcal{Y}/\mathcal{Z}}^{\operatorname{an}}$  by  $\mathbb{L}(n) := \tau_{[-n]} \mathbb{L}_{\mathcal{Y}/\mathcal{Z}}^{\operatorname{an}}$  and prove our assertion for the latter complexes for every integer  $n$ . Take a complex of  $B$ -flat modules of finite presented  $\mathbb{L}_0^\bullet$  which represents the complex  $\mathbb{L}(n)$ . Then we have  $(L\tilde{f}^*(\mathbb{L}(n))^{\operatorname{rig}} = (\mathbb{L}_0^\bullet \otimes_B A \otimes_R K)^\sim$  (By  $(\bullet)^\sim$  we denote the associated sheaf). On the other hand,  $\mathbb{L}_0^\bullet \otimes_R K$  is a complex of  $B \otimes_R K$ -flat module which represents  $\mathbb{L}(n)^{\operatorname{rig}}$ . Thus we have a quasi-isomorphism  $Lf^*\mathbb{L}(n) \cong ((\mathbb{L}_0^\bullet \otimes_R$

$K) \otimes_{(B \otimes K)} (A \otimes K))^\sim$ . Therefore, we have a natural quasi-isomorphism  $(\mathbf{L}(n) \overset{\mathbf{L}}{\otimes}_B A)^{\text{rig}} \xrightarrow{\sim} \mathbf{L}(n)^{\text{rig}} \overset{\mathbf{L}}{\otimes}_{(B \otimes_R K)} (A \otimes_R K)$ .  $\square$

Let  $f : X \rightarrow Y$  be a morphism of rigid analytic spaces over  $K$ . There exists a morphism of admissible formal  $R$ -schemes

$$\tilde{f} : \mathcal{X} \longrightarrow \mathcal{Y}$$

which is a formal model of  $f$  (cf. [5, Theorem 4.1]).

**Proposition 2.4.** *The complex  $(\mathbf{L}_{\mathcal{X}/\mathcal{Y}}^{\text{an}})^{\text{rig}}$  is independent of the choice of the formal model  $\tilde{f} : \mathcal{X} \rightarrow \mathcal{Y}$  of  $f : X \rightarrow Y$ , i.e., depends only on the morphism  $f : X \rightarrow Y$ .*

*Proof.* Let  $\tilde{f}' : \mathcal{X}' \rightarrow \mathcal{Y}'$  be another formal model of  $f : X \rightarrow Y$ . By an easy application of the theorem of Raynaud [5, Theorem 4.1], we can find the following commutative diagram

$$\begin{array}{ccccc} \mathcal{X} & \xleftarrow{\alpha} & \mathcal{X}'' & \xrightarrow{s} & \mathcal{X}' \\ \downarrow \tilde{f} & & \downarrow \tilde{f}'' & & \downarrow \tilde{f}' \\ \mathcal{Y} & \xleftarrow{\beta} & \mathcal{Y}'' & \xrightarrow{t} & \mathcal{Y} \end{array}$$

such that  $\tilde{f}, \tilde{f}', \tilde{f}''$  are formal models of  $f : X \rightarrow Y$  and  $\alpha, \beta, s, t$  are admissible blowing ups. It suffices to show that there are natural isomorphisms  $(\mathbf{L}_{\mathcal{X}/\mathcal{Y}}^{\text{an}})^{\text{rig}} \cong (\mathbf{L}_{\mathcal{X}''/\mathcal{Y}''}^{\text{an}})^{\text{rig}} \cong (\mathbf{L}_{\mathcal{X}'/\mathcal{Y}'}^{\text{an}})^{\text{rig}}$ . We will show an existence of the first isomorphism. The second one follows from the proof of the first one. To this end, consider the commutative diagram

$$\begin{array}{ccccc} \mathcal{X}'' & & & & \\ & \searrow \tilde{f}'' & & & \\ & & \mathcal{X} \times_{\mathcal{Y}} \mathcal{Y}'' & \xrightarrow{\text{pr}_2} & \mathcal{Y}'' \\ & \searrow \eta & \downarrow \text{pr}_1 & & \downarrow \beta \\ & & \mathcal{X} & \xrightarrow{f} & \mathcal{Y} \end{array}$$

where  $\mathcal{X} \times_{\mathcal{Y}} \mathcal{Y}''$  is the fiber product of  $\mathcal{X}$  and  $\mathcal{Y}''$  over  $\mathcal{Y}$  in the category of formal  $R$ -schemes and  $\eta$  is the induced morphism by  $\alpha$  and  $\tilde{f}''$ .

By the transitivity to the sequence  $\mathcal{X}'' \rightarrow \mathcal{X} \times_{\mathcal{Y}} \mathcal{Y}'' \rightarrow \mathcal{Y}''$  ([13, 7.2.13]), there exists a distinguished triangle

$$L\eta^* \mathbf{L}_{\mathcal{X} \times_{\mathcal{Y}} \mathcal{Y}''/\mathcal{Y}''}^{\text{an}} \rightarrow \mathbf{L}_{\mathcal{X}''/\mathcal{Y}''}^{\text{an}} \rightarrow \mathbf{L}_{\mathcal{X}''/\mathcal{X} \times_{\mathcal{Y}} \mathcal{Y}''}^{\text{an}} \rightarrow L\eta^* \mathbf{L}_{\mathcal{X} \times_{\mathcal{Y}} \mathcal{Y}''/\mathcal{Y}''}^{\text{an}}[1].$$

Now consider the functor of derived categories

$$\text{Rig} : \mathbf{D}^-(\mathcal{O}_{\mathcal{X}''}\text{-Mod}) \rightarrow \mathbf{D}^-(\mathcal{O}_X\text{-Mod}),$$

which is induced by  $M \mapsto (M)^{\text{rig}}$  for any  $\mathcal{O}''$ -module  $M$ . By applying this functor, we obtain the following distinguished triangle,

$$(L\eta^* \mathbf{L}_{\mathcal{X} \times_{\mathcal{Y}} \mathcal{Y}''/\mathcal{Y}''}^{\text{an}})^{\text{rig}} \rightarrow (\mathbf{L}_{\mathcal{X}''/\mathcal{Y}''}^{\text{an}})^{\text{rig}} \rightarrow (\mathbf{L}_{\mathcal{X}''/\mathcal{X} \times_{\mathcal{Y}} \mathcal{Y}''}^{\text{an}})^{\text{rig}} \rightarrow (L\eta^* \mathbf{L}_{\mathcal{X} \times_{\mathcal{Y}} \mathcal{Y}''/\mathcal{Y}''}^{\text{an}})^{\text{rig}}[1].$$

On the other hand, there is a natural quasi-isomorphism  $Lpr_1^* \mathbb{L}_{\mathcal{X}/\mathcal{Y}}^{\text{an}} \cong \mathbb{L}_{\mathcal{X} \times_{\mathcal{Y}} \mathcal{Y}''/\mathcal{Y}''}^{\text{an}}$  by Proposition 2.1. Hence we have  $L\alpha^* \mathbb{L}_{\mathcal{X}/\mathcal{Y}}^{\text{an}} \cong L\eta^* \mathbb{L}_{\mathcal{X} \times_{\mathcal{Y}} \mathcal{Y}''/\mathcal{Y}''}^{\text{an}}$  by Lemma 2.3 (1). Thus taking the above triangle into account, we see that if  $(\mathbb{L}_{\mathcal{X}''/\mathcal{X} \times_{\mathcal{Y}} \mathcal{Y}''})^{\text{rig}} = 0$ , there are isomorphisms,

$$(\mathbb{L}_{\mathcal{X}/\mathcal{Y}}^{\text{an}})^{\text{rig}} \cong (L\alpha^* \mathbb{L}_{\mathcal{X}/\mathcal{Y}}^{\text{an}})^{\text{rig}} \cong (L\eta^* \mathbb{L}_{\mathcal{X} \times_{\mathcal{Y}} \mathcal{Y}''/\mathcal{Y}''}^{\text{an}})^{\text{rig}} \cong (\mathbb{L}_{\mathcal{X}''/\mathcal{Y}''}^{\text{an}})^{\text{rig}}.$$

The first isomorphism in the above sequence follows from the fact that the functor  $(\bullet)^{\text{rig}}$  commutes with admissible blow-ups. Since  $\eta^{\text{rig}} : (\mathcal{X}'')^{\text{rig}} \rightarrow (\mathcal{X} \times_{\mathcal{Y}} \mathcal{Y}'')^{\text{rig}}$  induces the isomorphism of rigid analytic spaces, we have  $(\mathbb{L}_{\mathcal{X}''/\mathcal{X} \times_{\mathcal{Y}} \mathcal{Y}''}^{\text{an}})^{\text{rig}} = 0$  from [13, 7.2.42 (1)] and [7, Prop. 1.5].  $\square$

We define a rigid cotangent complex  $\mathbb{L}_{X/Y}^{\text{rig}}$  in  $\mathbf{D}^-(\mathcal{O}_X\text{-Mod})$  of  $f : X \rightarrow Y$  by  $(\mathbb{L}_{\mathcal{X}/\mathcal{Y}}^{\text{an}})^{\text{rig}}$  for some formal model  $\mathcal{X} \rightarrow \mathcal{Y}$ .

For every morphism  $\tilde{f} : \mathcal{X} \rightarrow \mathcal{Y}$  of admissible formal schemes, there is a natural morphism:  $\mathbb{L}_{\mathcal{X}/\mathcal{Y}}^{\text{an}} \rightarrow \Omega_{\mathcal{X}/\mathcal{Y}}^{\text{an}}$  which induces an isomorphism  $H_0(\mathbb{L}_{\mathcal{X}/\mathcal{Y}}^{\text{an}}) \cong \Omega_{\mathcal{X}/\mathcal{Y}}^{\text{an}}$  (cf. [13, 7.2.9]). Thus we have a natural morphism

$$\mathbb{L}_{X/Y}^{\text{rig}} \rightarrow \Omega_{X/Y}^{\text{rig}}$$

which induces an isomorphism

$$H_0(\mathbb{L}_{X/Y}^{\text{rig}}) \xrightarrow{\sim} \Omega_{X/Y}^{\text{rig}}$$

where  $X := \mathcal{X}^{\text{rig}}$  and  $Y := \mathcal{Y}^{\text{rig}}$  and  $\Omega_{X/Y}^{\text{rig}}$  is the differential module defined in [7, Section 1].

**Remark 2.5.** Our construction also works in the case of relative rigid spaces introduced and studied by Bosch, Lütkebohmert and Raynaud (cf. [5] [6] [7] [8]). However, in present paper, we concentrate our attentions on the classical case.

In virtue of the theorem of Raynaud [5, 4.1] and formal flattening theorem (See [6, 5.2]), as we will see below, the study of analytic cotangent complexes of rigid analytic spaces is reduced to the study of the formal case. Thus Proposition 2.4 is important.

**Proposition 2.6.** *Let  $f : X \rightarrow Y$  and  $g : Z \rightarrow Y$  be morphisms of rigid analytic spaces. Then there is a natural quasi-isomorphism*

$$Lpr_1^* \mathbb{L}_{X/Y}^{\text{rig}} \xrightarrow{\sim} \mathbb{L}_{X \times_Y Z/Z}^{\text{rig}}$$

where  $pr_1$  is the first projection  $X \times_Y Z \rightarrow X$ .

*Proof.* Take formal models  $\tilde{f} : \mathcal{X} \rightarrow \mathcal{Y}$  and  $\tilde{g} : \mathcal{Z} \rightarrow \mathcal{Y}$  of  $f$  and  $g$  respectively. Note that the fibre product  $\mathcal{X} \times_{\mathcal{Y}} \mathcal{Z}$  in the category of formal  $R$ -schemes is the formal model of  $X \times_Y Z$  by [5, 4.6]. Now our assertion follows from Proposition 2.1.  $\square$

**Proposition 2.7.** *If  $f : X \rightarrow Y$  is a smooth morphism, there is a natural quasi-isomorphism*

$$\mathbb{L}_{X/Y}^{\text{rig}} \xrightarrow{\sim} \Omega_{X/Y}^{\text{rig}}[0].$$

*Proof.* Take a formal model  $\tilde{f} : \mathcal{X} \rightarrow \mathcal{Y}$  of  $f : X \rightarrow Y$ . Then, we derive our assertion by applying [13, 7.2.42 (1)] to  $\tilde{f} : \mathcal{X} \rightarrow \mathcal{Y}$ .  $\square$

**Proposition 2.8.** *Let  $X \xrightarrow{f} Y \rightarrow Z$  be a sequence of morphisms of rigid analytic spaces. Then, there is a natural distinguished triangle in  $\mathbf{D}^-(\mathcal{O}_X\text{-Mod})$*

$$Lf^*\mathbb{L}_{Y/Z}^{\text{rig}} \rightarrow \mathbb{L}_{X/Z}^{\text{rig}} \rightarrow \mathbb{L}_{X/Y}^{\text{rig}} \rightarrow Lf^*\mathbb{L}_{Y/Z}^{\text{rig}}[1].$$

*Proof.* By the theorem of Raynaud [5, 4.1], there is a formal model of  $X \rightarrow Y \rightarrow Z$

$$\mathcal{X} \xrightarrow{\tilde{f}} \mathcal{Y} \rightarrow \mathcal{Z}.$$

From the transitivity of this sequence [13, 7.2.13], we have

$$L\tilde{f}^*\mathbb{L}_{\mathcal{Y}/\mathcal{Z}}^{\text{an}} \rightarrow \mathbb{L}_{\mathcal{X}/\mathcal{Z}}^{\text{an}} \rightarrow \mathbb{L}_{\mathcal{X}/\mathcal{Y}}^{\text{an}} \rightarrow L\tilde{f}^*\mathbb{L}_{\mathcal{Y}/\mathcal{Z}}^{\text{an}}[1].$$

By applying the functor of derived categories

$$\text{Rig} : \mathbf{D}^-(\mathcal{O}_{\mathcal{X}}\text{-Mod}) \rightarrow \mathbf{D}^-(\mathcal{O}_X\text{-Mod}),$$

we obtain the triangle

$$(L\tilde{f}^*\mathbb{L}_{\mathcal{Y}/\mathcal{Z}}^{\text{an}})^{\text{rig}} \rightarrow (\mathbb{L}_{\mathcal{X}/\mathcal{Z}}^{\text{an}})^{\text{rig}} \rightarrow (\mathbb{L}_{\mathcal{X}/\mathcal{Y}}^{\text{an}})^{\text{rig}} \rightarrow (L\tilde{f}^*\mathbb{L}_{\mathcal{Y}/\mathcal{Z}}^{\text{an}}[1])^{\text{rig}}.$$

Thus we have the required triangle by Lemma 2.3 (2).  $\square$

**Theorem 2.9.** (1) *Let  $i : Y \rightarrow X$  is a closed immersion of rigid analytic spaces. Then the natural morphism*

$$\mathbb{L}_{Y/X} \longrightarrow \mathbb{L}_{Y/X}^{\text{rig}}$$

*is a quasi-isomorphism. Here  $\mathbb{L}_{Y/X}$  is the usual cotangent complex associated to the morphism of ringed topoi  $i : Y \rightarrow X$ .*

(2) *Let  $f : X \rightarrow Y$  be a morphism of rigid analytic spaces. Then  $\mathbb{L}_{X/Y}^{\text{rig}}$  is a pseudo-coherent complex of  $\mathcal{O}_X$ -modules.*

*Proof.* (1) First of all, take a formal model  $\tilde{i} : \mathcal{Y} \rightarrow \mathcal{X}$  of  $i : Y \rightarrow X$ . Note that there is a natural isomorphism  $\mathbb{L}_{Y/X} \cong (\mathbb{L}_{\mathcal{Y}/\mathcal{X}})^{\text{rig}}$  by [17, 2.2.3]. Thanks to the formal flattening theorem [6, 5.4 (b)], we can modify  $\tilde{i} : \mathcal{Y} \rightarrow \mathcal{X}$  to be a closed immersion. Then the claim is reduced to [13, 7.2.10 (2)].

(2) Let  $\tilde{f} : \mathcal{X} \rightarrow \mathcal{Y}$  be a formal model of  $f : X \rightarrow Y$ . Then the claim is reduced to the claim for  $\tilde{f} : \mathcal{X} \rightarrow \mathcal{Y}$  [13, 7.2.10 (1)].  $\square$

**Proposition 2.10.** *Let  $f : X \rightarrow Y$  be a closed immersion of rigid analytic spaces and  $g : Y \rightarrow Z$  a smooth morphism of rigid analytic spaces. Let  $\mathcal{I}$  be a coherent ideal of  $\mathcal{O}_Y$  which defines  $X$ . Then there is a quasi-isomorphism*

$$\tau_{[-1]}\mathbb{L}_{X/Y}^{\text{rig}} \xrightarrow{\sim} [0 \rightarrow f^*(\mathcal{I}/\mathcal{I}^2) \xrightarrow{d} f^*\Omega_{Y/Z}^{\text{rig}} \rightarrow 0].$$

*Proof.* First, we remark that by [17, Chapter 3, 1.2.8.1] and Theorem 2.9 (1), there exists a natural quasi-isomorphisms,  $\tau_{[-1]} \mathbb{L}_{X/Y}^{\text{rig}} \cong \tau_{[-1]} \mathbb{L}_{X/Y} \cong f^*(\mathcal{I}/\mathcal{I}^2)[1]$ . Then, we can prove our assertion in the same way as the proof of [17, Chapter 3, 1.2.9.1] by using Propositions 2.7 and 2.8.  $\square$

Let  $X \rightarrow Y$  be a morphism of rigid analytic spaces and  $\mathcal{S}$  a coherent  $\mathcal{O}_X$ -module. Let us consider a triple  $(i : X \rightarrow X', \mathcal{S}, \phi)$  where  $i$  is a closed immersion of rigid analytic spaces over  $Y$  with the square-zero kernel  $\mathcal{I} = \text{Ker}(\mathcal{O}_{X'} \rightarrow \mathcal{O}_X)$  and an isomorphism of coherent  $\mathcal{O}_X$ -modules  $\phi : i^*\mathcal{I} \rightarrow \mathcal{S}$ . Then there exists a natural quasi-isomorphism  $\text{Ext}_{\mathcal{O}_X}^1(\mathbb{L}_{X/X'}^{\text{rig}}, \mathcal{S}) \cong \text{Hom}_{\mathcal{O}_X}(i^*\mathcal{I}, \mathcal{S})$ , since we have a natural isomorphism  $\mathbb{L}_{X/X'}^{\text{rig}} \cong i^*\mathcal{I}[1]$  by Proposition 2.10. On the other hand, the distinguished triangle (Proposition 2.8) associated to the sequence  $X \rightarrow X' \rightarrow Y$  induces a homomorphism  $p : \text{Ext}_{\mathcal{O}_X}^1(\mathbb{L}_{X/X'}^{\text{rig}}, \mathcal{S}) \rightarrow \text{Ext}_{\mathcal{O}_X}^1(\mathbb{L}_{X/Y}^{\text{rig}}, \mathcal{S})$ . Thus it gives rise to a map

$$e : \text{EX}_Y(X, \mathcal{S}) \longrightarrow \text{Ext}_{\mathcal{O}_X}^1(\mathbb{L}_{X/Y}^{\text{rig}}, \mathcal{S}),$$

$$(i : X \rightarrow X', \mathcal{S}, \phi) \mapsto p(\phi),$$

where  $\text{EX}_Y(X, \mathcal{S})$  is the set of isomorphism classes of a triple  $(i : X \rightarrow X', \mathcal{S}, \phi)$ . Note that the set  $\text{EX}_Y(X, \mathcal{S})$  is the subset of the set  $\text{EXal}_Y(X, \mathcal{S})$  of isomorphism classes of the extensions of  $X$  by  $\mathcal{S}$  over  $K$  as locally ringed spaces. Indeed for elements  $\alpha, \beta \in \text{EX}_Y(X, \mathcal{S})$  and an isomorphism  $\phi : \alpha \xrightarrow{\sim} \beta$  as locally ringed spaces over  $K$  which induces the identity on  $X$ ,  $\phi$  is actually the isomorphism as rigid analytic spaces by Proposition 1.6 and [3, 6.1.3.1]. Hence, taking [17, Chapter 3, 1.2.3] into account,  $e$  is injective because the composite map  $\text{EX}_Y(X, \mathcal{S}) \xrightarrow{e} \text{Ext}_{\mathcal{O}_X}^1(\mathbb{L}_{X/Y}^{\text{rig}}, \mathcal{S}) \rightarrow \text{Ext}_{\mathcal{O}_X}^1(\mathbb{L}_{X/Y}, \mathcal{S})$  is equal to the composite map  $\text{EX}_Y(X, \mathcal{S}) \rightarrow \text{EXal}_Y(X, \mathcal{S}) \xrightarrow{\sim} \text{Ext}_{\mathcal{O}_X}^1(\mathbb{L}_{X/Y}, \mathcal{S})$ .

**Theorem 2.11.** *The map  $e$  is a bijection.*

*Proof.* What we need to show is that elements of the image  $\text{Ext}^1(\Omega_{X/Y}^{\text{rig}}, \mathcal{S})$  in  $\text{Ext}^1(\Omega_{X/Y}^1, \mathcal{S}) \cong \text{EXal}_Y(X, \mathcal{S})$  represent the sheaves of rigid analytic spaces which are extensions of  $\mathcal{O}_X$  by  $\mathcal{S}$ . Therefore the problem is local on  $X$ . Set  $X := \text{Sp}(B)$  and  $Y := \text{Sp}(A)$ . We choose a closed immersion  $i : X \cong \text{Sp}(C/I) \rightarrow Z := \text{Sp}(C)$  where  $C = A\langle X_1, \dots, X_r \rangle$ . By Proposition 2.10, we have

$$\text{Ext}_{\mathcal{O}_X}^1(\mathbb{L}_{X/Y}^{\text{rig}}, \mathcal{S}) \cong \text{Hom}(I/I^2, \mathcal{S})/\text{Image}(d^*).$$

However a map  $\phi : I/I^2 \rightarrow \mathcal{S}$  and the canonical immersion  $j : I/I^2 \rightarrow C/I^2$  define an extension  $(C/I^2 \oplus \mathcal{S})/\text{Image}((j, \phi))^\sim$  of  $\mathcal{O}_X$  by  $\mathcal{S}$  which is a sheaf arising from an affinoid algebra. It is easy to see that this extension corresponds to the element  $\phi$  in  $\text{Ext}_{\mathcal{O}_X}^1(\mathbb{L}_{X/Y}^{\text{rig}}, \mathcal{S})$ .  $\square$



## 3. COHOMOLOGICAL DESCRIPTIONS OF LOCAL DEFORMATIONS

Now by applying the results in section 2, we will prove the following theorem.

**Theorem 3.1.** *Let  $u : S \rightarrow S'$  be a closed immersion of rigid analytic spaces with the nilpotent kernel  $\mathcal{I}$  of  $u^* : \mathcal{O}_{S'} \rightarrow \mathcal{O}_S$  such that  $\mathcal{I}^2 = 0$ . Let  $f : X \rightarrow S$  be a flat morphism of rigid analytic spaces over  $K$ .*

- (1) *The obstruction for the existence of the lifting of  $f$  to  $S'$  lies in  $\mathrm{Ext}_{\mathcal{O}_X}^2(\mathbb{L}_{X/S}^{\mathrm{rig}}, f^*\mathcal{I})$ .*
- (2) *If the obstruction  $o$  is zero, the set of isomorphism classes of the liftings of  $f$  to  $S'$  forms a torsor under  $\mathrm{Ext}_{\mathcal{O}_X}^1(\mathbb{L}_{X/S}^{\mathrm{rig}}, f^*\mathcal{I})$ .*
- (3) *Let  $\tilde{f} : \tilde{X} \rightarrow S'$  be a flat deformation of  $X$ . Then, the automorphism group of the lifting  $\tilde{X}$  is canonically isomorphic to  $\mathrm{Ext}_{\mathcal{O}_X}^0(\mathbb{L}_{X/S}^{\mathrm{rig}}, f^*\mathcal{I})$ .*

*Proof.* By applying Proposition 2.8 to the sequence  $X \rightarrow S \rightarrow S'$ , we have a distinguished triangle

$$Lf^*\mathbb{L}_{S/S'}^{\mathrm{rig}} \rightarrow \mathbb{L}_{X/S}^{\mathrm{rig}} \rightarrow \mathbb{L}_{X/S'}^{\mathrm{rig}} \rightarrow Lf^*\mathbb{L}_{S/S'}^{\mathrm{rig}}[1].$$

On the other hand, by Proposition 2.10, we have  $\mathbb{L}_{S/S'}^{\mathrm{rig}} \cong u^*\mathcal{I}[1]$ . Thus there exists a natural isomorphism,  $\mathrm{Ext}_{\mathcal{O}_X}^1(Lf^*\mathbb{L}_{S/S'}^{\mathrm{rig}}, \mathcal{I}\mathcal{O}_X) \cong \mathrm{Hom}_{\mathcal{O}_X}(f^*\mathcal{I}\mathcal{O}_S, \mathcal{I}\mathcal{O}_X)$ . Since  $\Omega_{S/S'}^{\mathrm{rig}} = 0$ , we have  $\mathrm{Ext}_{\mathcal{O}_X}^0(Lf^*\mathbb{L}_{S/S'}^{\mathrm{rig}}, \mathcal{I}\mathcal{O}_X) \cong \mathrm{Hom}_{\mathcal{O}_X}(f^*\Omega_{S/S'}^{\mathrm{rig}}, \mathcal{I}\mathcal{O}_X) = 0$ . Thus there is a long exact sequence

$$\begin{array}{ccccccc} 0 & \longrightarrow & \mathrm{Ext}_{\mathcal{O}_X}^1(\mathbb{L}_{X/S}^{\mathrm{rig}}, \mathcal{I}\mathcal{O}_X) & \longrightarrow & \mathrm{Ext}_{\mathcal{O}_X}^1(\mathbb{L}_{X/S'}^{\mathrm{rig}}, \mathcal{I}\mathcal{O}_X) & & \\ & & \searrow_{\xi} & & \searrow_{\delta} & & \\ & & \mathrm{Hom}_{\mathcal{O}_X}(f^*\mathcal{I}\mathcal{O}_S, \mathcal{I}\mathcal{O}_X) & \longrightarrow & \mathrm{Ext}_{\mathcal{O}_X}^2(\mathbb{L}_{X/S}^{\mathrm{rig}}, \mathcal{I}\mathcal{O}_X) & & \end{array}$$

An element in  $\mathrm{EX}_{S'}(X, \mathcal{I}\mathcal{O}_X)$  canonically induces an element which lies in  $\mathrm{Hom}_{\mathcal{O}_X}(f^*\mathcal{I}\mathcal{O}_S, \mathcal{I}\mathcal{O}_X)$ . Thus we have a map

$$\alpha : \mathrm{EX}_{S'}(X, \mathcal{I}\mathcal{O}_X) \rightarrow \mathrm{Hom}_{\mathcal{O}_X}(f^*\mathcal{I}\mathcal{O}_S, \mathcal{I}\mathcal{O}_X).$$

There is a natural isomorphism  $\mathrm{EX}_{S'}(X, \mathcal{I}\mathcal{O}_X) \xrightarrow{\sim} \mathrm{Ext}^1(\mathbb{L}_{X/S'}^{\mathrm{rig}}, \mathcal{I}\mathcal{O}_X)$  by Theorem 2.11 and this isomorphism identifies  $\xi$  with  $\alpha$ . Let  $u$  be the element in  $\mathrm{Hom}_{\mathcal{O}_X}(f^*\mathcal{I}\mathcal{O}_S, \mathcal{I}\mathcal{O}_X)$  which is induced by  $f$ . The existence of a flat deformation of  $X$  over  $S'$  is equivalent to the existence of an element of  $\mathrm{Ext}_{\mathcal{O}_X}^1(\mathbb{L}_{X/S'}^{\mathrm{rig}}, \mathcal{I}\mathcal{O}_X)$  which induces  $u$  by  $\xi$ , so (1) follows. Now it is clear that if  $\delta u = 0$ , the isomorphism classes of flat deformations is  $\xi^{-1}(u) \cong \mathrm{Ext}_{\mathcal{O}_X}^1(\mathbb{L}_{X/S}^{\mathrm{rig}}, \mathcal{I}\mathcal{O}_X)$ . This shows (2). Finally the isomorphism  $H_0(\mathbb{L}_{X/Y}^{\mathrm{rig}}) \cong \Omega_{X/Y}^{\mathrm{rig}}$  implies that  $\mathrm{Ext}_{\mathcal{O}_X}^0(\mathbb{L}_{X/S}^{\mathrm{rig}}, \mathcal{I}\mathcal{O}_X) \cong \mathrm{Hom}(\Omega_{X/S}^{\mathrm{rig}}, \mathcal{I}\mathcal{O}_X)$ . It is well-known that for every flat deformation of  $X$  to  $S'$ , its automorphism group is isomorphic to the right-hand group. (We can show this by the completely same argument as the scheme-case (cf. [14]))  $\square$

## 4. FORMAL VERSAL DEFORMATION

**4.1. Preliminaries.** In this section, we will consider the deformations of rigid analytic  $K$ -spaces to the spectrums of local Artin  $K$ -algebras. Since every local Artin  $K$ -algebra is a finite  $K$ -vector space, it is the topological ring whose topology are induced by the topology of  $K$ . We call this topology *the canonical topology*.

**Lemma 4.1.** *Let  $K$  be a complete non-Archimedean valued field. Then, an Artin local  $K$ -algebra with the canonical topology is a  $K$ -affinoid algebra.*

*Proof.* Since a local Artin  $K$ -algebra  $A$  is of finite type over  $K$ , we have the representation of  $A$  by the quotient ring of a polynomial ring

$$A \cong K[X_1, \dots, X_n]/I.$$

If we equip  $A$  with canonical topology and  $K[X_1, \dots, X_n]/I$  with Gauss norm, this is an isomorphism of complete topological rings. The Tate algebra  $K\langle X_1, \dots, X_n \rangle$  is flat over  $K[X_1, \dots, X_n]$ . Thus there exists a canonical isomorphism

$$K[X_1, \dots, X_n]/I \cong K\langle X_1, \dots, X_n \rangle / IK\langle X_1, \dots, X_n \rangle.$$

This completes the proof.  $\square$

**Remark 4.2.** By the above lemma, we can attach an affinoid space  $\mathrm{Sp}(A)$  to any Artin local  $K$ -algebra  $A$ . Since every homomorphism of affinoid algebras as  $K$ -algebras is automatically a continuous homomorphism, we can view the category of Artin local  $K$ -algebras as the full subcategory of the category of  $K$ -affinoid algebras.

**Definition 4.3.** For a rigid analytic space  $X$  over  $K$ , a *local deformation functor*  $\mathbb{D}_X$  of  $X$  is the functor defined as follows:

$$\mathbb{D}_X : \left( \text{Artin local } K\text{-algebras with residue field } K \right) \longrightarrow \left( \text{Sets} \right)$$

Let  $A$  be an Artin local  $K$ -algebra with the maximal ideal  $\mathfrak{m}_A$  and the residue field  $K$ . Then the set  $\mathbb{D}_X(A)$  is the isomorphism classes of pairs  $(f : \tilde{X} \rightarrow \mathrm{Sp}(A), \alpha : \tilde{X} \times_A A/\mathfrak{m}_A \xrightarrow{\sim} X)$  of a flat morphism  $f$  of rigid analytic spaces over  $K$  and  $\alpha$  is an isomorphism of rigid analytic spaces over  $K$ .

**4.2. Schlessinger's theory.** Let  $X$  be a rigid analytic spaces over  $\mathrm{Sp}(K)$ .

**Definition 4.4.** Let  $\mathcal{O}$  be a complete local noetherian  $K$ -algebra with maximal ideal  $\mathfrak{m}$  with residue field  $K$ . Let  $\{X_n\}_{n \geq 0}$  be a family of deformations of  $X$  to  $\mathrm{Sp}(\mathcal{O}/\mathfrak{m}^{n+1})$  such that  $X_n$  is a flat deformation to  $\mathrm{Sp}(\mathcal{O}/\mathfrak{m}^{n+1})$  and  $X_n \times_{\mathrm{Sp}(\mathcal{O}/\mathfrak{m}^{n+1})} \mathrm{Sp}(\mathcal{O}/\mathfrak{m}^{m+1}) \cong X_m$  for  $m \leq n$ . We say that a pair  $(\mathcal{O}, \{X_n\}_{n \geq 0})$  is a *formal versal deformation* of  $X$  if it satisfies the following conditions.

- (1) Suppose that  $A$  is an Artin local  $K$ -algebra with the maximal ideal  $\mathfrak{m}_A$  such that  $\mathfrak{m}_A^{n+1} = 0$ . If  $\tilde{X}$  is a deformation of  $X$  to  $\mathrm{Sp}(A)$ , there exists a local homomorphism  $f : \mathcal{O}/\mathfrak{m}^{n+1} \rightarrow A$  such that  $\tilde{X}$  is isomorphic to  $X_n \times_{\mathrm{Sp}(\mathcal{O}/\mathfrak{m}^{n+1})} \mathrm{Sp}(A)$ .
- (2) If  $A = K[\epsilon]/(\epsilon^2)$ , such  $f$  is unique.

The existence of a formal versal deformation of  $X$  is equivalent to the existence of a prorepresentational hull of the functor  $\mathbb{D}_X$ , in the sense of Schlessinger (cf. [21]).

We have a convenient criterion for the existence of a prorepresentational hull.

**Theorem 4.5** (Schlessinger [21]). *Let  $A' \rightarrow A$  and  $A'' \rightarrow A$  be morphisms of Artin local  $K$ -algebras. Consider the natural map*

$$F : \mathbb{D}_X(A' \times_A A'') \rightarrow \mathbb{D}_X(A') \times_{\mathbb{D}_X(A)} \mathbb{D}_X(A'').$$

*The  $\mathbb{D}_X$  has a prorepresentational hull if and only if the following conditions are satisfied:*

- (H1) *If  $A'' \rightarrow A$  is a small surjection,  $F$  is surjective.*
- (H2)  *$F$  is bijective when  $A = K$ ,  $A'' = K[\epsilon]/(\epsilon^2)$ .*
- (H3)  *$\dim_K \mathbb{D}_X(K[\epsilon]/(\epsilon^2)) < \infty$ .*

#### 4.3. Existence of a formal versal deformation.

**Theorem 4.6.** *Let  $X$  be a proper rigid analytic space over  $K$ . Then, there exists a formal versal deformation of  $X$ .*

First we will show (H1).

**Claim 4.6.1.** *The functor  $\mathbb{D}_X$  satisfies (H1).*

*Proof.* Let  $(\xi', \xi'')$  be an element of  $\mathbb{D}_X(A') \times_{\mathbb{D}_X(A)} \mathbb{D}_X(A'')$ . We put

$$\begin{aligned} \xi &:= (\tilde{X}/\mathrm{Sp}(A), \phi : \tilde{X} \times_A K \cong X), \\ \xi' &:= (\tilde{X}'/\mathrm{Sp}(A'), \phi' : \tilde{X}' \times_A K \cong X), \\ \xi'' &:= (\tilde{X}''/\mathrm{Sp}(A''), \phi : \tilde{X}'' \times_A K \cong X), \end{aligned}$$

such that  $\alpha^* \xi' = \xi$  and  $\beta^* \xi'' = \xi$  where  $A' \xrightarrow{\alpha} A$ ,  $A'' \xrightarrow{\beta} A$ . Fix an affinoid open set  $\mathrm{Sp}(R)$  of  $\tilde{X}$ . Note that  $|X| = |\tilde{X}| = |\tilde{X}'| = |\tilde{X}''|$ . Here  $|\bullet|$  means the underlying set. The subspace  $\tilde{X}'|_{\mathrm{Sp}(R)}$  (resp.  $\tilde{X}''|_{\mathrm{Sp}(R)}$ ) of  $\tilde{X}'$  (resp.  $\tilde{X}''$ ) which is the lifting of  $\mathrm{Sp}(R)$  is an affinoid space by Proposition 1.6. We set  $\mathrm{Sp}(R') = \tilde{X}'|_{\mathrm{Sp}(R)}$ ,  $\mathrm{Sp}(R'') = \tilde{X}''|_{\mathrm{Sp}(R)}$  and  $s : R' \rightarrow R$ ,  $t : R'' \rightarrow R$ . To show our claim, it suffices to prove that  $R' \times_R R''$  is an affinoid algebra. First we define a norm on  $R' \times_R R''$  by the direct sum of the norms of  $R'$  and  $R''$ . This norm induces the topology on  $R' \times_R R''$ . This topology is complete. Indeed, if  $\{(a_n, b_n)\}_{n \leq 1}$  is a Cauchy sequence of  $R' \times_R R''$ ,  $\{s(a_n) = t(b_n)\}_{n \leq 1}$  is also a Cauchy sequence of  $R$  since  $|t(b_n) - t(b_m)| \leq |b_n - b_m|$ . Now what to do is to construct a continuous surjective map from an affinoid algebra to  $R' \times_R R''$ . To this end, take a generator  $\{\xi_1, \dots, \xi_r\}$  of  $\mathrm{Ker}(t)$  such

that  $|\xi_i|_{sp} \leq 1$  for  $1 \leq i \leq r$ . Next put  $R' = K\langle X_1, \dots, X_n \rangle / I$ . We can choose elements  $\eta_1, \dots, \eta_n$  in  $R''$  such that  $s(X_i) = t(\eta_i)$  and  $|\eta_i|_{sp} \leq 1$  for  $1 \leq i \leq n$ . Indeed, since  $t$  is a small surjection,  $|r|_{sp} = |t(r)|_{sp}$  for  $r \in R''$  and  $|s(X_i)|_{sp} \leq |X_i|_{sp} \leq |X_i| \leq 1$  for  $1 \leq i \leq n$ . Therefore, we have the following homomorphism

$$K[S_1, \dots, S_n, T_1, \dots, T_r] \rightarrow R' \times_R R''$$

defined by  $S_i \rightarrow (X_i, \eta_i)$  and  $T_j \rightarrow (0, \xi_j)$ . Since  $|X_i|_{sp} \leq 1$ ,  $|\eta_i|_{sp} \leq 1$  and  $|\xi_j|_{sp} \leq 1$  for all  $i$  and  $j$ , this homomorphism is uniquely extended to the continuous homomorphism (See [12, 3.4.7])

$$K\langle S_1, \dots, S_n, T_1, \dots, T_r \rangle \rightarrow R' \times_R R''.$$

Furthermore, from the construction, it is clear that this continuous homomorphism is surjective. Hence we see (H1).  $\square$

Next we show (H2) and (H3).

**Claim 4.6.2.** *The functor  $\mathbb{D}_X$  satisfies (H2).*

*Proof.* We can show this by the completely same argument as the scheme-case (cf. [21]).  $\square$

**Claim 4.6.3.** *The functor  $\mathbb{D}_X$  satisfies (H3).*

*Proof.* Let  $f : X \rightarrow \mathrm{Sp}(K)$  be a structure morphism. Then, the isomorphism of derived functors

$$Rf_* R\mathcal{H}om(-, \mathcal{O}_X) \cong R\mathcal{H}om(-, \mathcal{O}_X)$$

induces the spectral sequence

$$E_2^{p,q} = R^q f_* \mathcal{E}xt^p(\mathcal{L}_{X/K}^{\mathrm{rig}}, \mathcal{O}_X) \Rightarrow \mathrm{Ext}^{p+q}(\mathcal{L}_{X/K}^{\mathrm{rig}}, \mathcal{O}_X).$$

Then it suffices to show that  $R^q f_* \mathcal{E}xt^p(\mathcal{L}_{X/K}^{\mathrm{rig}}, \mathcal{O}_X)$  is coherent for all  $p$  and  $q$ . Note that by Theorem 2.9 (2),  $\mathcal{L}_{X/K}^{\mathrm{rig}}$  is pseudo-coherent. Thus  $R\mathcal{H}om(\mathcal{L}_{X/K}^{\mathrm{rig}}, \mathcal{O}_X)$  is pseudo-coherent by [11, Chapter 0, 12.3.3] and [16, Chapter 1, 7.3]. By Kiehl's finiteness theorem,  $R^q f_* \mathcal{E}xt^p(\mathcal{L}_{X/K}^{\mathrm{rig}}, \mathcal{O}_X)$  is a finite  $K$ -vector space for all  $p$  and  $q$ .  $\square$

By the above three claims, we can complete the proof of Theorem 4.6 by Schlessinger's criterion.

## 5. TOWARDS THE GLOBAL MODULI THEORY VIA RIGID GEOMETRY

In Section 4, we proved the existence of a versal family in the formal sense for deformations of rigid analytic spaces. However the analogy of rigid geometry with complex analytic geometry gives us a deeper question. When one compares the deformation theory of complex analytic spaces with one of the algebraic categories, no one doubts that the most important advantage is the existence of versal family of deformations proven by Kuranishi and Grauert (cf. [15] [19]). Formal deformations of algebraic varieties are not

necessarily algebraic. Thus there is no algebraic analogue of the theorem of Kuranishi and Grauert. As we know, there is no logical relation between complex geometry and rigid geometry. However one can conjecture the following.

**Conjecture.** Let  $X$  be a proper rigid analytic spaces over  $K$ . Then there exists a flat morphism of rigid analytic spaces  $F : \mathfrak{X} \longrightarrow \mathfrak{S}$  and a  $K$ -rational point  $p$  of  $\mathfrak{S}$  such that the completion of  $F$  at  $p$  is isomorphic to the formal versal deformation of  $X$ .

This assertion can be viewed as a fairly precise non-Archimedean analogue of the existence theorem of versal families for deformations of complex analytic spaces due to Kuranishi and Grauert. Let  $\mathcal{S}$  be a rigid analytic torus, i.e., a quotient space  $T^{\text{an}}/\Gamma$  where  $T$  is a split  $K$ -torus and  $\Gamma$  is a torsion-free lattice of rank  $\dim T$ . Then, we can prove that the conjecture holds for  $\mathcal{S}$  by using  $p$ -adic uniformization theory due to Bosch-Lütkebohmert-Raynaud. Unfortunately, at the time of writing this paper, the author do not have a proof of this conjecture for general rigid analytic spaces. But the author expects that this conjecture is true and propose it.

Let us give one sufficient condition which implies the conjecture. Let  $K$  be a discrete valuation field and  $R$  its ring of integers with residue field  $k$ . Let  $X$  be a proper rigid analytic  $K$ -space. To prove the conjecture for  $X$ , it suffices to show the existence of a locally noetherian adic formal scheme over  $\text{Spf } R$  which satisfies the followings (cf. [2] section 0.2).

- (1) Its reduction is a scheme locally of finite type over  $\text{Spec } k$ .
- (2) It is a formal model of the formal deformation family of a rigid analytic spaces  $X$ .

However it seems difficult to prove the conjecture in general.

Let us explain why this problem is important. Suppose that we want to construct a moduli space of interesting geometric objects. From the stack theoretic viewpoint, versal spaces for deformations of them (here “versal space” is not in formal sense but has a geometric structure such as a scheme (resp. complex analytic or rigid analytic etc...)) are local components of the smooth cover of an algebraic (resp. complex analytic, rigid analytic) stack (cf. [1]). Thus, roughly speaking, this conjecture says that in rigid geometry, the existence of local deformation theory implies the global moduli stack which is represented by a rigid analytic stack.

Furthermore this conjecture will be important to the geometry of algebraic schemes over an arbitrary field. Let  $X$  be an algebraic scheme over an arbitrary field  $k$ . By considering the constant deformation  $\hat{X} \rightarrow \text{Spf}(k[[T]])$ , one can associate a rigid analytic space  $\hat{X}^{\text{rig}} \rightarrow \text{Spf}(k[[T]])^{\text{rig}}$ . By this technique the deformation of  $X$  to a family of algebraic schemes is contained in the rigid analytic versal family if it exists. Therefore the above conjecture is of prime importance not only in rigid geometry but also in algebraic geometry.

## APPENDIX

In this appendix, we prove a convenient criterion of the existence of a non-trivial square-zero extension of a ringed topos.

Let  $A$  be a ring and  $B$  an  $A$ -algebra. Let  $M$  be a  $B$ -module and  $\mathbb{L}_{B/A}$  the cotangent complex of the structure homomorphism  $A \rightarrow B$ . By  $\mathrm{EXal}_A(B, M)$ , we denote the set of isomorphism classes of square-zero extension of  $B$  by  $M$  over  $A$ . By the fundamental theorem due to L. Illusie ([17, Chapter 3, 1.2.3]), there exists a natural bijection

$$\phi : \mathrm{Ext}_A^1(\mathbb{L}_{B/A}, M) \xrightarrow{\sim} \mathrm{EXal}_A(B, M).$$

On the other hand, there exists a natural homomorphism

$$\pi : \mathrm{Ext}_A^1(\Omega_{B/A}^1, M) \rightarrow \mathrm{Ext}_A^1(\mathbb{L}_{B/A}, M)$$

which is induced by  $\mathbb{L}_{B/A} \rightarrow \Omega_{B/A}^1$  (cf. [17, Chapter 2, 1.2.4]).

**Theorem A.1.** *The natural map  $\pi : \mathrm{Ext}_A^1(\Omega_{B/A}^1, M) \rightarrow \mathrm{Ext}_A^1(\mathbb{L}_{B/A}, M)$  is injective. In particular, if  $\mathrm{Ext}_A^1(\Omega_{B/A}^1, M) \neq 0$ , there exists a non-trivial square-zero extension of  $B$  by  $M$  over  $A$ .*

*Proof.* First, note that it suffices to show that the map  $\psi := \phi \circ \pi : \mathrm{Ext}_A^1(\Omega_{B/A}^1, M) \rightarrow \mathrm{EXal}_A(B, M)$  is injective. Let us construct explicitly the map  $\psi$  (cf. [17, Chapter 3, 1.1.8]). For an element  $\xi$  in  $\mathrm{Ext}_A^1(\Omega_{B/A}^1, M)$ , let  $(0 \rightarrow M \xrightarrow{\alpha} N \xrightarrow{\beta} \Omega_{B/A}^1 \rightarrow 0)$  be the corresponding short exact sequence. We define an  $A$ -algebra structure of  $B \oplus N$  by  $(b, n) + (b', n') := (b + b', n + n')$  and  $(b, n) \cdot (b', n') := (b \cdot b', b \cdot n' + b' \cdot n)$ . We also define an  $A$ -algebra structure of  $B \oplus \Omega_{B/A}^1$  by the same way. Consider the following diagram

$$\begin{array}{ccccccc} 0 & \longrightarrow & M & \longrightarrow & C & \xrightarrow{\mathrm{pr}_2} & B \longrightarrow 0 \\ & & \downarrow \mathrm{Id} & & \downarrow \mathrm{pr}_1 & & \downarrow (\mathrm{Id}, d_{B/A}) \\ 0 & \longrightarrow & M & \xrightarrow{(0, \alpha)} & B \oplus N & \xrightarrow{(\mathrm{Id}, \beta)} & B \oplus \Omega_{B/A}^1 \longrightarrow 0, \end{array}$$

where  $C$  is the fibre product  $(B \oplus N) \times_{(B \oplus \Omega_{B/A}^1)} B$ . Then we define  $\psi(\xi)$  by  $(0 \rightarrow M \rightarrow C \rightarrow B \rightarrow 0)$ . Since  $\mathrm{EXal}_A(B, M)$  has a group structure by  $\phi$ , it suffices to prove the following claim.

**Claim A.1.1.** *Under the same assumption as above, if the exact sequence  $(0 \rightarrow M \rightarrow C \rightarrow B \rightarrow 0)$  has a splitting  $a : B \rightarrow C$ , the exact sequence  $(0 \rightarrow M \rightarrow N \rightarrow \Omega_{B/A}^1 \rightarrow 0)$  also has a splitting  $\Omega_{B/A}^1 \rightarrow N$ .*

*Proof.* Let  $\mathrm{Der}_A(B, N)$  be the set of  $A$ -derivations of  $B$  to  $N$ . Then note that we have a natural isomorphisms  $\mathrm{Hom}_B(\Omega_{B/A}^1, N) \xrightarrow{\sim} \mathrm{Der}_A(B, N)$  and  $\mathrm{Der}_A(B, N) \xrightarrow{\sim} \mathrm{Hom}_{A\text{-alg}/B}(B, B \oplus N)$  (cf. [17, Chapter 2, 1.1.1.4 and 1.1.2.6]). Let  $\delta : \Omega_{B/A}^1 \rightarrow N$  be a homomorphism that corresponds to the

element  $\mathrm{pr}_1 \circ a$  in  $\mathrm{Hom}_{A\text{-alg}/B}(B, B \oplus N)$  by the above isomorphisms. Then it is easy to see that  $\delta$  is a splitting of  $\beta : N \rightarrow \Omega_{B/A}^1 \rightarrow 0$ .  $\square$

Thus we complete the proof of the theorem.  $\square$

Let  $f : X \rightarrow Y$  be a morphism of ringed topoi. Let  $\Omega_{X/Y}^1$  be a Kähler differential module of  $f^{-1}(\mathcal{O}_Y) \rightarrow \mathcal{O}_X$ , and  $\mathbb{L}_{X/Y}$  the cotangent complex (cf. [17, chapter 2]). By  $\mathrm{EXal}_Y(X, \mathcal{M})$ , we denote the set of isomorphism classes of square-zero extension of  $X$  by a  $\mathcal{O}_X$ -module  $\mathcal{M}$  over  $Y$ . By the same procedure with the above, we have a natural bijection  $\phi : \mathrm{Ext}_Y^1(\mathbb{L}_{X/Y}, \mathcal{M}) \xrightarrow{\sim} \mathrm{EXal}_Y(X, \mathcal{M})$  and a homomorphism  $\pi : \mathrm{Ext}_Y^1(\Omega_{X/Y}^1, \mathcal{M}) \rightarrow \mathrm{Ext}_Y^1(\mathbb{L}_{X/Y}, \mathcal{M})$ .

**Corollary A.2.** *Let  $X \rightarrow Y$  be a morphism of ringed topoi and  $\mathcal{M}$  a  $\mathcal{O}_X$ -module. The natural map  $\pi : \mathrm{Ext}_Y^1(\Omega_{X/Y}^1, \mathcal{M}) \rightarrow \mathrm{Ext}_Y^1(\mathbb{L}_{X/Y}, \mathcal{M})$  is injective. In particular, if  $\mathrm{Ext}_Y^1(\Omega_{X/Y}^1, \mathcal{M}) \neq 0$ , there exists a non-trivial square-zero extension of  $X$  by  $\mathcal{M}$  over  $Y$ .*

## REFERENCES

- [1] M. Artin, Versal deformations and Algebraic stacks, Invent. Math. vol. 27 (1974) 165–184
- [2] P. Berthelot, Cohomologie rigide et cohomologie rigide support propres, Prépublication IRMAR 96-03, 89 pages (1996).
- [3] S. Bosch, U. Guntzer and R. Remmert, Non-Archimedean analysis. A systematic approach to rigid analytic geometry. Grundlehren der math. Wiss., 261 Springer-Verlag, Berlin, 1984.
- [4] S. Bosch and W. Lütkebohmert, Degenerating abelian varieties, Topology, vol. 30 No. 4, (1991) 653–698
- [5] S. Bosch and W. Lütkebohmert. Formal and Rigid Geometry I. Math. Ann, 295 (1993), no.2, 291–317.
- [6] S. Bosch and W. Lütkebohmert. Formal and Rigid Geometry II. Math. Ann, 296 (1993), no.3, 403–429.
- [7] S. Bosch, W. Lütkebohmert and M. Raynaud. Formal and Rigid Geometry III. Math. Ann, 302 (1995), no.1, 1–29.
- [8] S. Bosch, W. Lütkebohmert and M. Raynaud. Formal and Rigid Geometry IV. Invent. Math., 119 (1995), no.2, 361–398.
- [9] N. Bourbaki, Elément de Mathématique: Algèbre Commutative, Pris: Hermann 1961–1965
- [10] J. Diouddonné and A. Grothendieck, Éléments de Géométrie Algébriques I, Publ. Math. IHES 4 (1960)
- [11] J. Diouddonné and A. Grothendieck, Éléments de Géométrie Algébriques Chapitre III, Publ. Math. IHES 11 (1961)
- [12] J. Fresnel and M. van der Put. Rigid Analytic Geometry and Its Applications. Progress in Math. vol.218 (2003)
- [13] O. Gabber and L. Ramero, Almost ring theory, Lecture Note in Math. 1800 (2003)
- [14] A. Grothendieck, Revêtement Étale et Groupe Fondamental, Lecture Note in Math. 224 Springer-Verlag, Berlin, 1971
- [15] H. Grauert, Der Satz von Kuranishi für komplexe Räume, Invent. Math. 25 (1974) 107–142
- [16] R. Harshorne, Residues and duality. Lecture Note in Math. 20 (1966)
- [17] L. Illusie, Complexe Cotangent et Déformations 1 Lecture Note in Math. 239 (1971)

- [18] M. Kashiwara and P. Scapira, Sheaves on manifolds, Die Grundlehren der Math. Wiss., 292 (1990) Springer
- [19] M. Kuranishi, On the locally complete families of complex analytic structures, Ann. Math. (2) vol. 75, No. 3 (1962) 536–577
- [20] M. Rapoport and T. Zink, Period spaces for  $p$ -divisible groups. Annals of Math. Studies. Princeton University Press. (1996)
- [21] M. Schlessinger, Functor of Artin rings, Trans. Amer. Soc. 130 (1968) 208–222

DEPARTMENT OF MATHEMATICS, GRADUATE SCHOOL OF SCIENCE, KYOTO UNIVERSITY, KYOTO, 606-8502, JAPAN

*E-mail address:* `iwanari@math.kyoto-u.ac.jp`